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# A FULL DISCRETIZATION OF THE TIME-DEPENDENT NAVIER-STOKES EQUATIONS BY A TWO-GRID SCHEME

HYAM ABBOUD[†,‡] AND TONI SAYAH[‡]

Abstract. We study a two-grid scheme fully discrete in time and space for solving the Navier-Stokes system. In the first step, the fully non-linear problem is discretized in space on a coarse grid with mesh-size $H$ and time step $k$. In the second step, the problem is discretized in space on a fine grid with mesh-size $h$ and the same time step, and linearized around the velocity $u_H$ computed in the first step. The two-grid strategy is motivated by the fact that under suitable assumptions, the contribution of $u_H$ to the error in the non-linear term, is measured in the $L^2$ norm in space and time, and thus has a higher-order than if it were measured in the $H^1$ norm in space. We present the following results: if $h = H^2 = k$, then the global error of the two-grid algorithm is of the order of $h$, the same as would have been obtained if the non-linear problem had been solved directly on the fine grid.

**Keywords** Two-grid scheme, Non-linear problem, Incompressible flow, Time and Space discretizations, Duality argument, "superconvergence".

## 1. Introduction.

The two-grid method is a general strategy for solving a non-linear Partial Differential Equation (PDE), depending or not in time, with solution $u$. In a first step, we discretize the fully non-linear PDE on a coarse grid of mesh-size $H$ and we compute an approximate solution $u_H$. Then, in a second step, we linearize the PDE around $u_H$ and we discretize the linearized problem on a fine grid of mesh-size $h$; let $u_h^{lin}$ be the corresponding solution. Then, under suitable assumptions, we can prove that if $h, H$ and the time step $k$ are well-chosen, the global error of the two-grid algorithm $\| u - u_h^{lin} \|$ has the same order as the error $\| u - u_h \|$ that would have been obtained if the non-linear problem had been directly discretized on the fine grid.

Two-grid discretizations have been widely applied to linear and non-linear elliptic boundary value problems: J. Xu in [20], [21], [22] has pioneered their development. These methods have been extended to the steady Navier-Stokes equations, cf. for instance the work of W. Layton in [11], W. Layton & W. Lenferink in [12] and V. Girault & J.-L. Lions in [5]. Also, this method has been applied to the time-dependent Navier-Stokes problem, cf. V. Girault & J.-L. Lions [6] in which they analyze a semi-discrete algorithm.
The purpose of this article is to solve by a two-grid scheme, on a coarse grid and a fine grid, the non-stationary incompressible Navier-Stokes problem and to show that the two-grid algorithm's global error is similar to the error of the direct resolution of the non-linear problem on a fine grid.

Let $\Omega$ be a bounded domain of $\mathbb{R}^2$ with a polygonal boundary $\partial\Omega$ and let $]0, T[$ be a given time-interval. Consider the following Navier-Stokes equations for an incompressible fluid, with $u$ the velocity and $p$ the pressure

$$\frac{\partial u}{\partial t}(x,t) - \nu\Delta u(x,t) + u(x,t)\cdot\nabla u(x,t) + \nabla p(x,t) = f(x,t) \text{ in } \Omega\times]0,T[, \tag{1.1}$$

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with the incompressibility condition

$$\operatorname{div} u(x,t) = 0 \text{ in } \Omega\times]0,T[, \tag{1.2}$$

the homogeneous Dirichlet boundary condition

$$u(x,t) = 0 \text{ on } \partial\Omega\times]0,T[, \tag{1.3}$$

and the initial condition

$$u(x,0) = 0 \text{ in } \Omega, \tag{1.4}$$

where the notation $u \cdot \nabla u$ means

$$u \cdot \nabla u = \sum_{i=1}^{2} u_i \frac{\partial u}{\partial x_i}.$$

Setting $L_0^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q \; dx = 0\}$ and assuming that $f$ belongs to $L^2(0,T;H^{-1}(\Omega)^2)$, it is well-known that (1.1)–(1.2) has the following variational formulation in $]0,T[$:

Find $u(t) \in H_0^1(\Omega)^2$, such that in the sense of distributions on $]0,T[$,

$$\forall v \in H_0^1(\Omega)^2, \frac{d}{dt}(u(t),v) + \nu(\nabla u(t), \nabla v) + (u(t)\cdot\nabla u(t), v) - (p(t), \operatorname{div} v) = \langle f(t), v\rangle, \tag{1.5}$$

$$\forall q \in L_0^2(\Omega), (q, \operatorname{div} u(t)) = 0, \tag{1.6}$$

and

$$u(0) = 0, \tag{1.7}$$

where $u(t) = u(x,t)$.

Furthermore, this problem has one and only one solution $u$ in $L^\infty(0,T;L^2(\Omega)^2) \cap L^2(0,T;H^1(\Omega)^2)$ and $p$ in the dual space of $W_0^{1,1}(0,T;L_0^2(\Omega))$ (see e.g. O.A. Ladyzenskaya in [10] and J.-L. Lions in [13]). In addition, we have the following regularity result.

**Theorem 1.1.** *If $\Omega$ is convex and $f \in L^2(0,T;L^2(\Omega)^2)$, then*

$$u \in L^\infty(0,T;H^1(\Omega)^2) \cap L^2(0,T;H^2(\Omega)^2) \text{ and } p \in L^2(0,T;H^1(\Omega)). \tag{1.8}$$

For discretizing (1.5)–(1.7), let $\eta > 0$ be a discretization parameter in space and for each $\eta$, let $\mathcal{T}_\eta$ be a corresponding regular (or non-degenerate) family of triangulations of $\overline{\Omega}$, consisting of triangles such that any two triangles are either disjoint or share a vertex or an entire side. For an arbitrary triangle $\kappa$, we denote by $\eta_\kappa$ the diameter of $\kappa$ and by $\rho_\kappa$ the diameter of the circle inscribed in $\kappa$. Then $\eta$ denotes the maximum of $\eta_\kappa$ and we assume that $\mathcal{T}_\eta$ is regular in the sense of Ciarlet [4] : there exists a constant $\sigma$ independent of $\eta$ such that

$$\sup_{\kappa\in\mathcal{T}_\eta} \frac{\eta_\kappa}{\rho_\kappa} = \sigma_\kappa \leq \sigma. \tag{1.9}$$

Let $X_\eta$ and $M_\eta$ be a "stable" pair of finite-element spaces for discretizing the velocity $u$ and the pressure $p$, stable in the sense that it satisfies a uniform discrete inf-sup condition: there exists a constant $\beta^\star \geq 0$, independent of $\eta$, such that

$$\forall q_\eta \in M_\eta, \sup_{v_\eta\in X_\eta} \frac{1}{|v_\eta|_{H^1(\Omega)}} \int_\Omega q_\eta \operatorname{div} v_\eta dx \geq \beta^\star \parallel q_\eta \parallel_{L^2(\Omega)} . \tag{1.10}$$

Let $\mathbb{P}_\kappa$ denote the space of polynomials with total degree less than or equal to $\kappa$. As the two-grid scheme is better adapted to finite-elements of low degree, we may choose for instance the "mini-element" (see D. Arnold, F. Brezzi and M. Fortin in [3]), where in each triangle $\kappa$, the pressure $p$ is a polynomial of $\mathbb{P}_1$ and each component of the velocity is the sum of a polynomial of $\mathbb{P}_1$ and a "bubble" function $b_\kappa$.

Denoting the vertices of $\kappa$ by $a_i, 1 \le i \le 3$, and its corresponding barycentric coordinate by $\lambda_i$, the basic bubble function $b_\kappa$ is the polynomial of degree three

$$b_\kappa(x) = \lambda_1(x)\lambda_2(x)\lambda_3(x).$$

We observe that $b_\kappa(x) = 0$ on $\partial\kappa$ and that $b_\kappa(x) > 0$ on $\kappa$. The graph of $b_\kappa$ looks like a bulb attached to the boundary of $\kappa$, whence its name.
Therefore, the finite-element spaces are :

$$X_\eta = \left\{ v_\eta \in C^0(\overline{\Omega})^2;\ \forall \kappa \in \mathcal{T}_\eta,\ v_{\eta|_\kappa} \in \mathcal{P}(\kappa), v_{\eta|\partial\Omega} = 0 \right\}, \tag{1.11}$$

$$M_\eta = \left\{ q_\eta \in C^0(\overline{\Omega});\ \forall \kappa \in \mathcal{T}_\eta,\ q_{\eta|_\kappa} \in \mathbb{P}_1, \int_\Omega q_\eta dx = 0 \right\}, \tag{1.12}$$

where

$$\mathcal{P}(\kappa) = [\mathbb{P}_1 \oplus Vect(b_\kappa)]^2. \tag{1.13}$$

There exists an approximation operator $P_\eta \in \mathcal{L}(H_0^1(\Omega)^2; X_\eta)$ such that (see V. Girault and P.-A. Raviart in [7]) :

$$\forall v \in H_0^1(\Omega)^2,\ \ \forall q_\eta \in M_\eta,\ \ \int_\Omega q_\eta \operatorname{div}(P_\eta(v) - v)dx = 0, \tag{1.14}$$

and for $k = 0$ or 1,

$$\forall v \in [H^{1+k}(\Omega) \cap H_0^1(\Omega)]^2,\ \ \| P_\eta(v) - v \|_{L^2(\Omega)} \le\ C\eta^{1+k}|v|_{H^{1+k}(\Omega)}, \tag{1.15}$$

and forall $r \ge 2, k = 0$ or 1,

$$\forall v \in [W^{1+k,r}(\Omega) \cap H_0^1(\Omega)]^2,\ \ |P_\eta(v) - v|_{W^{1,r}(\Omega)}\ \le\ C_r\eta^k|v|_{W^{1+k,r}(\Omega)}. \tag{1.16}$$

In addition, as $M_\eta$ contains all polynomials of degree one, there exists an operator $r_\eta \in \mathcal{L}(L_0^2(\Omega); M_\eta)$, such that for any real number $s \in [0, 2]$,

$$\forall q \in H^s(\Omega) \cap L_0^2(\Omega), \| r_\eta(q) - q \|_{L^2(\Omega)} \le C\eta^s|q|_{H^s(\Omega)}. \tag{1.17}$$

To discretize in time, we divide the interval $[0, T]$ into $N$ subintervals of equal length $k = \dfrac{T}{N}$, with grid-points $t^n = nk, 0 \le n \le N$.

With these spaces, we propose the following two-grid scheme for discretizing (1.5)–(1.7). We use two regular triangulations of $\overline{\Omega}$ : a coarse triangulation $\mathcal{T}_H$ and a fine one $\mathcal{T}_h$, that for practical purposes, is a refinement of $\mathcal{T}_H$.
On each of these, we define the same stable pair of finite-element spaces, $(X_H, M_H)$ and $(X_h, M_h)$ such that $X_H \subset X_h$ and $M_H \subset M_h$. At each time step, we solve (1.18)–(1.19) and (1.20)–(1.21) below. The two-grid algorithm reads :

• **Step One** (non-linear problem on coarse grid): Knowing $u_H^n$, find $(u_H^{n+1}, p_H^{n+1})$ with values in $X_H \times M_H$, solution of

$$\forall v_H \in X_H, \frac{1}{k}(u_H^{n+1} - u_H^n, v_H)\ +\ \nu(\nabla u_H^{n+1}, \nabla v_H)\ +\ (u_H^{n+1} \cdot \nabla u_H^{n+1}, v_H) + \frac{1}{2}(\operatorname{div} u_H^{n+1}, u_H^{n+1} \cdot v_H)$$
$$- (p_H^{n+1}, \operatorname{div} v_H) = \langle f^{n+1}, v_H \rangle \tag{1.18}$$

$$\forall q_H \in M_H,\ (q_H, \operatorname{div} u_H^{n+1}) = 0. \tag{1.19}$$

• **Step Two** (linearized problem on fine grid): Knowing $(u_H^{n+1}, p_H^{n+1})$, find $(u_h^{n+1}, p_h^{n+1})$ with values in $X_h \times M_h$ solution of

$$\forall v_h \in X_h, \frac{1}{k}(u_h^{n+1} - u_h^n, v_h) \ + \ \nu(\nabla u_h^{n+1}, \nabla v_h) \ + \ (u_H^{n+1} \cdot \nabla u_h^{n+1}, v_h) \ - \ (p_h^{n+1}, \operatorname{div} v_h) = \langle f^{n+1}, v_h \rangle \tag{1.20}$$

$$\forall q_h \in M_h, \ (q_h, \operatorname{div} u_h^{n+1}) = 0. \tag{1.21}$$

By assumption, $u_h^0 = 0$. Moreover, at the step time $n+1$, in (1.18), $u_H^n$ is in fact a restriction on the coarse grid of $u_h^n$ that has just been computed :

$$u_H^n = \mathcal{R}(u_h^n),$$

where $\mathcal{R}$ is a suitable restriction from $X_h$ into $X_H$.
The pressure $p_h^{n+1}$ is dissociated from $u_h^{n+1}$ by a decoupling algorithm starting with an extension of $p_H^{n+1}$ to the fine grid.
In both (1.18) and (1.20), $f^{n+1}$ is a suitable approximation of $f$ at time $t^{n+1}$. The purpose of this two-grid algorithm is to reduce the time of computation for both velocity and pressure.

In the sequel, we shall take $k$ of the order of $H^2$ : there exist two constants $\alpha_1$ and $\alpha_2 > 0$ that do not depend on $H$ and $k$ such that

$$\alpha_1 H^2 \leq k \leq \alpha_2 H^2.$$

**Remark 1.2.** To simplify the error analysis, the convection term in (1.18) is stabilized so that it is anti-symmetric. But often in practice, it is not stabilized; it is the case of the numerical tests presented at the end. We refer to [6] for the numerical analysis of a semi-discrete scheme that is not stabilized.

**Remark 1.3.** One can also linearize the first step by taking the non-linear term at time $n$ (instead of $n+1$). This requires a condition CFL, but as $k << H$, this condition is generally satisfied.

**Remark 1.4.** This is an example in which both equations use the same time step and are both of order one with respect to time. A more elaborate idea for Step Two would be to use a scheme of second-order in time with the same time step, or some time-splitting scheme of order one.

The remainder of this article is organized as follows: In Section 2, we present some conventions and notations that will be used throughout the article. In Section 3, we present a first error estimate for the fully-discrete Step One then in section 4 we establish a duality argument based on the backward semi-discrete Stokes system and we derive some uniform bounds that allow us to prove the Stokes problem's error estimate in $L^2(\Omega\times]0, T[)^2$, then we apply it to the Navier-Stokes problem. We also prove a "superconvergence" result for the non-linear part. The pressure is estimated in section 5 and the error estimation for the solution of Step Two is studied in section 6. Finally, in section 7, we confirm these results numerically.
Some of these results have been announced in [1].

## 2. Preliminaries.

To begin with, we present some conventions and notations that will be used throughout the article. As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $]a, b[$ with values in a functional space, say $X$ (cf. Lions and Magenes [14]). More precisely, let $\| \cdot \|_X$ denote the norm of $X$; then for any $r$, $1 \leq r \leq \infty$, we define

$$L^r(a, b; X) = \left\{ f \text{ mesurable in } ]a, b[; \int_a^b \| \ f(t) \ \|_X^r \ dt < \infty \right\}$$

equipped with the norm

$$\| f \|_{L^r(a,b;X)}= \Big( \int_a^b \| f(t) \|_X^r \, dt \Big)^{1/r},$$

with the usual modifications if $r=\infty$. It is a Banach space if $X$ is a Banach space.

Let $(k_1,k_2)$ denote a pair of non-negative integers, set $|k|=k_1+k_2$ and define the partial derivative $\partial^k$ by

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} \partial x_2^{k_2}}.$$

Here $X$ is usually a Sobolev space, such as (cf. Adams [2] or Nečas [15]): for any non-negative integer $m$ and number $r \ge 1$,

$$W^{m,r}(\Omega) = \{ v \in L^r(\Omega); \partial^k v \in L^r(\Omega), \forall |k| \le m \}.$$

This space is equipped with the seminorm

$$|v|_{W^{m,r}(\Omega)} = \Big[ \sum_{|k|=m} \int_\Omega |\partial^k v|^r dx \Big]^{1/r},$$

and is a Banach space for the norm

$$\| v \|_{W^{m,r}(\Omega)}= \Big[ \sum_{0 \le |k| \le m} |v|^r_{W^{k,r}(\Omega)} \Big]^{1/r},$$

with the usual extension when $r=\infty$.
When $r=2$, this space is the Hilbert space $H^m(\Omega)$. In particular, the scalar product of $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$.

Similarly, $L^2(a,b;H^m(\Omega))$ is a Hilbert space and in particular $L^2(a,b;L^2(\Omega))$ coincides with $L^2(\Omega\times]a,b[)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation, but with the following modification for the norms in the non-Hilbert case. Let $u=(u_1,u_2)$; then we set

$$\| u \|_{L^r(\Omega)}= \Big[ \int_\Omega \| u(x) \|^r \, dx \Big]^{1/r},$$

where $\| \cdot \|$ denotes the Euclidean vector norm.

For functions that vanish on the boundary, we define for any $r \ge 1$ :

$$W_0^{1,r}(\Omega) = \{ v \in W^{1,r}(\Omega); v|_{\partial\Omega} = 0 \}, \tag{2.1}$$

and recall Poincaré's inequality: there exists a constant $\mathcal{P}$ such that

$$\forall v \in H_0^1(\Omega), \| v \|_{L^2(\Omega)} \le \mathcal{P} |v|_{H^1(\Omega)}. \tag{2.2}$$

More generally, recall the inequalities of Sobolev imbeddings in two dimensions: for each $r \in [2,\infty[$, there exits a constant $S_r$ such that

$$\forall v \in H_0^1(\Omega) \ , \ \| v \|_{L^r(\Omega)} \le S_r |v|_{H^1(\Omega)}, \tag{2.3}$$

where

$$|v|_{H^1(\Omega)} = \| \nabla v \|_{L^2(\Omega)} \ . \tag{2.4}$$

When $r=2$, (2.3) reduces to Poincaré's inequality and $S_2$ is Poincaré's constant.
The case $r=\infty$ is excluded and is replaced by: for any $r>2$, there exists a constant $M_r$ such that

$$\forall v \in W_0^{1,r}(\Omega) \ , \ \| v \|_{L^\infty(\Omega)} \le M_r |v|_{W^{1,r}(\Omega)}. \tag{2.5}$$

We have also in dimension 2,

$$\| g \|_{L^4(\Omega)} \leq 2^{1/4} \| g \|_{L^2(\Omega)}^{1/2} \| \nabla g \|_{L^2(\Omega)}^{1/2} . \tag{2.6}$$

Owing to (2.2), we use the seminorm $|\cdot|_{H^1(\Omega)}$ as a norm on $H_0^1(\Omega)$ and we use it to define the norm of the dual space $H^{-1}(\Omega)$ of $H_0^1(\Omega)$:

$$\| f \|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v \rangle}{|v|_{H^1(\Omega)}},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Also, we recall the spaces we introduced at the beginning:

$$V = \{v \in H_0^1(\Omega)^2; \operatorname{div} v = 0 \text{ in } \Omega\},$$

$$L_0^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q \, dx = 0\},$$

and the orthogonal complement of $V$ in $H_0^1(\Omega)^2$ :

$$V^\perp = \{v \in H_0^1(\Omega)^2; \forall w \in V, (\nabla v, \nabla w) = 0\}.$$

## 3. Error estimates for the solution of Step One

The results in this paragraph are written for the non-linear scheme (1.18)–(1.19).
To simplify, we denote by $\eta$ the mesh parameter. The first result, stated in Lemma 3.1, is a standard error estimate. We give the proof for the sake of completeness.

**Lemma 3.1.** *Let $X_\eta$ and $M_\eta$ be defined by* (1.11) *and* (1.12) *and approximate $f^{n+1}$ by the average defined almost everywhere in $\Omega$ as follows :*

$$f^{n+1}(x) = \frac{1}{k} \int_{t^n}^{t^{n+1}} f(x,t) dt, \quad a.e \ \ x \in \Omega. \tag{3.1}$$

*At each time step,* (1.18)–(1.19) *has a solution $u_\eta^{n+1}$ and this solution is unique if $k$ is sufficiently small. If $f$ and $u$ are sufficiently smooth, each solution satisfies :*

$$\sup_{0 \leq n \leq N} \| u_\eta^n - u(t^n) \|_{L^2(\Omega)} + \Big( \sum_{n=0}^{N-1} \| (u_\eta^{n+1} - u(t^{n+1})) - (u_\eta^n - u(t^n)) \|_{L^2(\Omega)}^2 \Big)^{1/2}$$
$$+ \sqrt{\nu} \Big( \sum_{n=0}^{N-1} k |u_\eta^{n+1} - u(t^{n+1})|_{H^1(\Omega)}^2 \Big)^{1/2} \leq C(f,u,p,\nu,T)(\eta + k), \tag{3.2}$$

*with a constant $C(f,u,p,\nu,T)$ independent of $\eta$ and $k$.*

*Proof.* Noting that the approximation operator $P_\eta$ defined in [7] satisfies $(P_\eta(u))' = P_\eta(u')$, we integrate (1.1) over $[t^n, t^{n+1}]$, substract (1.18), insert $P_\eta u(t^{n+1})$, choose the test function $v_\eta^{n+1} = u_\eta^{n+1} - P_\eta u(t^{n+1})$, multiply the result by the time step $k$ and sum it over $n = 0, ..., m-1$. We obtain :

$$\begin{aligned} &\frac{1}{2}(\| v_\eta^m \|_{L^2(\Omega)}^2 - \| v_\eta^0 \|_{L^2(\Omega)}^2 + \sum_{n=0}^{m-1} \| v_\eta^{n+1} - v_\eta^n \|_{L^2(\Omega)}^2) + \nu \sum_{n=0}^{m-1} k \| \nabla v_\eta^{n+1} \|_{L^2(\Omega)}^2 \\ &= \sum_{n=0}^{m-1} \Big\{ ((u(t^{n+1}) - P_\eta u(t^{n+1})) - (u(t^n) - P_\eta u(t^n)), v_\eta^{n+1}) + \nu \int_{t^n}^{t^{n+1}} (\nabla(u(t) - P_\eta u(t^{n+1})), \nabla v_\eta^{n+1}) dt \\ &+ \int_{t^n}^{t^{n+1}} \Big( (u(t) \cdot \nabla u(t) - u_\eta^{n+1} \cdot \nabla u_\eta^{n+1}) + \frac{1}{2}(\operatorname{div} u(t) u(t) - \operatorname{div} u_\eta^{n+1} u_\eta^{n+1}), v_\eta^{n+1} \Big) dt \\ &- \int_{t^n}^{t^{n+1}} (p(t) - r_\eta p(t), \operatorname{div} v_\eta^{n+1}) dt \Big\}. \end{aligned} \tag{3.3}$$

Let us study the terms of the right hand side of (3.3). The non-linear term is treated like follows :

$$\begin{aligned}u(t)\cdot\nabla u(t)-u_\eta^{n+1}\cdot\nabla u_\eta^{n+1} &= (u(t)-P_\eta u(t^{n+1}))\cdot\nabla u(t)-v_\eta^{n+1}\cdot\nabla P_\eta u(t^{n+1})\\ &+ P_\eta u(t^{n+1})\cdot\nabla(u(t)-P_\eta u(t^{n+1}))+u_\eta^{n+1}\cdot\nabla(P_\eta u(t^{n+1})-u_\eta^{n+1})\end{aligned} \tag{3.4}$$

and the term corresponding to the divergence is treated similarly.

The first term is bounded as follows : For any $\varepsilon_1>0$,

$$\begin{aligned}&\Big|\sum_{n=0}^{m-1}((u(t^{n+1})-P_\eta u(t^{n+1}))-(u(t^n)-P_\eta u(t^n)),v_\eta^{n+1})\Big|\\ &\leq\frac{C^2}{2\varepsilon_1}\parallel u'\parallel^2_{L^2(0,T;H^1(\Omega)^2)}\eta^2+\frac{\varepsilon_1 S_2}{2}\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)},\end{aligned}$$

where $S_2$ is the constant of Poincaré's inequality.

To study the second term, we insert $P_\eta u(t)$ and we obtain two terms : For any $\varepsilon_2>0$, the first one is bounded as follows :

$$\begin{aligned}&\Big|\nu\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}(\nabla(u(t)-P_\eta u(t)),\nabla v_\eta^{n+1})dt\Big|\\ &\leq\frac{\nu}{2}\Big\{\frac{1}{\varepsilon_2}\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}|u(t)-P_\eta u(t)|^2_{H^1(\Omega)}dt+\varepsilon_2\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)}\Big\}\\ &\leq\frac{\nu}{2}\Big\{\frac{C}{\varepsilon_2}\parallel u\parallel^2_{L^2(0,T;H^2(\Omega)^2)}\eta^2+\varepsilon_2\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)}\Big\},\end{aligned}$$

and the second one as follows : Knowing that

$$\int_{t^n}^{t^{n+1}}P_\eta(u(t)-u(t^{n+1}))dt=\int_{t^n}^{t^{n+1}}P_\eta u'(\tau)(\tau-t^n)d\tau,$$

we have, for any $\varepsilon_3>0$,

$$\Big|\nu\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}(\nabla P_\eta(u(t)-u(t^{n+1})),\nabla v_\eta^{n+1})dt\Big|\leq\frac{\nu C}{2\sqrt{3}\varepsilon_3}\parallel u'\parallel^2_{L^2(0,T;H^1(\Omega)^2)}k^2+\frac{\nu}{2\sqrt{3}}\varepsilon_3\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)}.$$

For the pressure contribution, we have, for any $\varepsilon_4>0$,

$$\begin{aligned}\Big|\sum_{n=0}^{m-1}-\int_{t^n}^{t^{n+1}}(p(t)-r_\eta p(t),\operatorname{div}v_\eta^{n+1})dt\Big| &\leq(\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\parallel p(t)-r_\eta p(t)\parallel^2_{L^2(\Omega)}dt)^{1/2}(\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)})^{1/2}\\ &\leq\frac{C}{2\varepsilon_4}\parallel p\parallel^2_{L^2(0,T;H^1(\Omega))}\eta^2+\frac{\varepsilon_4}{2}\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)}.\end{aligned}$$

Now, we consider the non-linear terms. Applying (2.3) and (2.6) and setting

$$C_1=\parallel u\parallel_{L^\infty(0,T;H^1(\Omega)^2)},$$

we have, for any $\varepsilon_5$ and $\varepsilon_6>0$,

$$\Big|\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}((u(t)-P_\eta u(t))\cdot\nabla u(t),v_\eta^{n+1})dt\Big|\leq\frac{C_1S_4^2}{2}\Big\{\frac{C}{\varepsilon_5}\parallel u\parallel^2_{L^2(0,T;H^2(\Omega)^2)}\eta^2+\varepsilon_5\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)}\Big\},$$

and

$$\Big|\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}(P_\eta(u(t)-u(t^{n+1}))\cdot\nabla u(t),v_\eta^{n+1})dt\Big|\leq\frac{C_1S_4^2}{2\sqrt{3}}\Big\{\frac{k^2}{\varepsilon_6}\parallel u'\parallel^2_{L^2(0,T;H^1(\Omega)^2)}+\varepsilon_6\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)}\Big\}.$$

The corresponding divergence terms are bounded as follows : For any $\varepsilon_7$ and $\varepsilon_8>0$,

$$\Big|\frac{1}{2}\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}(\mathrm{div}(u(t)-P_\eta u(t))\cdot u(t),v_\eta^{n+1})dt\Big|\leq\frac{S_4^2C_1}{4}\Big\{\frac{C}{\varepsilon_7}\parallel u\parallel^2_{L^2(0,T;H^2(\Omega)^2)}\eta^2+\varepsilon_7\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)}\Big\},$$

and

$$\Big|\frac{1}{2}\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}(\mathrm{div}\,P_\eta(u(t)-u(t^{n+1}))\cdot u(t),v_\eta^{n+1})dt\Big|\leq\frac{S_4^2C_1}{4\sqrt{3}}\Big\{\frac{k^2}{\varepsilon_8}\parallel u'\parallel^2_{L^2(0,T;H^1(\Omega)^2)}+\varepsilon_8\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)}\Big\}.$$

Setting $C_2=\parallel P_\eta u\parallel_{L^\infty(0,T;H^1(\Omega)^2)}$, we also have, for any $\varepsilon_9$ and $\varepsilon_{10}>0$,

$$\Big|-\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}(v_\eta^{n+1}\cdot\nabla P_\eta u(t^{n+1}),v_\eta^{n+1})dt\Big|\leq\frac{2^{1/2}C_1}{2}\Big\{\varepsilon_9\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)}+\frac{1}{\varepsilon_9}\sum_{n=0}^{m-1}k\parallel v_\eta^{n+1}\parallel^2_{L^2(\Omega)}\Big\},$$

and

$$\Big|-\frac{1}{2}\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}(\mathrm{div}\,v_\eta^{n+1}\cdot P_\eta u(t^{n+1}),v_\eta^{n+1})dt\Big|\leq\frac{2^{1/4}S_4C_2}{4}\Big\{\frac{1}{\varepsilon_{10}}\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)}$$
$$+\frac{\varepsilon_{10}}{2}\sum_{n=0}^{m-1}k(\delta|v_\eta^{n+1}|^2_{H^1(\Omega)}+\frac{1}{\delta}\parallel v_\eta^{n+1}\parallel^2_{L^2(\Omega)})\Big\}.$$

And the two final terms are split as follows : For any $\varepsilon_{11}>0$,

$$\Big|\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}(P_\eta u(t^{n+1})\cdot\nabla(u(t)-P_\eta u(t)),v_\eta^{n+1})dt\Big|\leq\frac{S_4^2C_2}{2}\Big\{\frac{C}{\varepsilon_{11}}\parallel u\parallel^2_{L^2(0,T;H^2(\Omega)^2)}\eta^2$$
$$+\varepsilon_{11}\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)}\Big\},$$

with the divergence contribution : For any $\varepsilon_{12}$ and $\varepsilon_{13}>0$,

$$\Big|\frac{1}{2}\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}(\mathrm{div}(P_\eta u(t^{n+1}))(u(t)-P_\eta u(t)),v_\eta^{n+1})dt\Big|\leq\frac{S_4^2C_1}{2}\Big\{\frac{C}{\varepsilon_{12}}\parallel u\parallel^2_{L^2(0,T;H^2(\Omega)^2)}\eta^4$$
$$+\varepsilon_{12}\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)}\Big\},$$

and

$$\Big|\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}(P_\eta u(t^{n+1})\cdot\nabla P_\eta(u(t)-u(t^{n+1})),v_\eta^{n+1})dt\Big|\leq\frac{S_4^2C_2C}{2\sqrt{3}}\Big\{\frac{k^2}{\varepsilon_{13}}\parallel u'\parallel^2_{L^2(0,T;H^1(\Omega)^2)}$$
$$+\varepsilon_{13}\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)}\Big\},$$

and also the divergence contribution : For any $\varepsilon_{14}>0$,

$$\Big|\frac{1}{2}\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}(\mathrm{div}(P_\eta u(t^{n+1}))P_\eta(u(t)-u(t^{n+1})),v_\eta^{n+1})dt\Big|\leq\frac{S_4^2C_2C}{4\sqrt{3}}\Big\{\frac{k^2}{\varepsilon_{14}}\parallel u'\parallel^2_{L^2(0,T;H^1(\Omega)^2)}$$
$$+\varepsilon_{14}\sum_{n=0}^{m-1}k|v_\eta^{n+1}|^2_{H^1(\Omega)}\Big\}.$$

After a suitable choice of $\varepsilon_i$ and $\delta$, (3.3) becomes

$$\frac{1}{2}\parallel v_\eta^m \parallel_{L^2(\Omega)}^2 + \frac{1}{2}\sum_{n=0}^{m-1}\parallel v_\eta^{n+1}-v_\eta^n\parallel_{L^2(\Omega)}^2 + \frac{\nu}{2}\sum_{n=0}^{m-1} k|v_\eta^{n+1}|_{H^1(\Omega)}^2 \le C_\star + C\sum_{n=0}^{m-1} k\parallel v_\eta^{n+1}\parallel_{L^2(\Omega)}^2$$

where $C_\star = \alpha\eta^2+\beta k^2, \alpha$ and $\beta$ are constants that depend on $u, p, \nu$, but do not depend on $\eta$ and $k$.

Then after applying Gronwall's lemma and for $k$ sufficiently small, the result follows from this inequality:

$$\sup_{0\le n\le N}\parallel u_\eta^n - P_\eta u(t^n)\parallel_{L^2(\Omega)} + \Big(\sum_{n=0}^{N-1}\parallel (u_\eta^{n+1}-P_\eta u(t^{n+1})) - (u_\eta^n - P_\eta u(t^n))\parallel_{L^2(\Omega)}^2\Big)^{1/2}$$

$$+\sqrt{\nu}\Big(\sum_{n=0}^{N-1} k|u_\eta^{n+1}-P_\eta u(t^{n+1})|_{H^1(\Omega)}^2\Big)^{1/2} \le C(\eta+k).$$

Finally, (3.2) follows by applying a triangular inequality and the $P_\eta$'s properties. □

The next property of the solution of (1.18)–(1.19) is an easy consequence of Lemma 3.1.

**Corollary 3.2.** *In addition to the assumptions of Lemma* 3.1, *we assume that there exist constants* $\alpha>0$ *and* $C$ *independent of* $\eta$ *and* $k$, *such that* $k\ge\alpha\eta^2$ *and*

$$\sup_n |u_\eta^n|_{H^1(\Omega)} \le C. \tag{3.5}$$

*Proof.* We have

$$\Big(\sum_{n=0}^{N-1} k|u_\eta^{n+1}-u(t^{n+1})|_{H^1(\Omega)}^2\Big)^{1/2} \le C(\eta+k),$$

which implies that

$$|u_\eta^n - u(t^n)|_{H^1(\Omega)}^2 \le \frac{C(\eta+k)^2}{k} \le C(\frac{\eta^2}{k}+k) \le C, \quad 0\le n\le N.$$

Then

$$|u_\eta^n|_{H^1(\Omega)} \le |u_\eta^n - u(t^n)|_{H^1(\Omega)} + |u(t^n)|_{H^1(\Omega)} \le C.$$

□

**Remark 3.3.** *We suppose that there exist two constants* $\alpha$ *and* $\gamma>0$ *that do not depend on* $\eta$ *and* $k$ *such that*

$$\alpha\eta^2 \le k \le \gamma\eta^2, \tag{3.6}$$

*which means that* $k$ *is of the same order of* $\eta^2$.

## 4. Some error estimates for the Stokes problem

The error estimate of order two in $L^2(\Omega\times]0,T[)^2$, that will be established in the next section, is based on a duality argument for the transient Stokes problem :

$$\frac{\partial v}{\partial t}(x,t) - \nu\Delta v(x,t) + \nabla q(x,t) = g(x,t) \text{ in } \Omega\times]0,T[, \tag{4.1}$$

$$\operatorname{div} v(x,t) = 0 \text{ in } \Omega\times]0,T[, \tag{4.2}$$

$$v(x,t) = 0 \text{ on } \partial\Omega\times]0,T[, \tag{4.3}$$

$$v(x,0) = 0 \text{ in } \Omega. \tag{4.4}$$

**Theorem 4.1.** *This problem has a unique solution $(v,q)$. We assume that $g \in L^2(\Omega\times]0,T[)^2$; Then*

$$v \in L^2(0,T;W^{2,4/3}(\Omega)^2) \cap L^\infty(0,T;H^1(\Omega)^2),\ v' \in L^2(\Omega\times]0,T[)^2 \ \text{ and } \ q \in L^2(0,T;W^{1,4/3}(\Omega)).$$

*If $\Omega$ is convex, then $(v,q) \in L^2(0,T;H^2(\Omega)^2) \times L^2(0,T;H^1(\Omega))$. Finally, without convexity assumption, if $g \in H^1(0,T;H^{-1}(\Omega)^2)$ and $g(0) \in L^2(\Omega)^2$, then $v' \in L^\infty(0,T;L^2(\Omega)^2) \cap L^2(0,T;H^1(\Omega)^2)$.*

The fully-discrete scheme for (4.1)–(4.4) is : Find $(v_\eta^{n+1}, q_\eta^{n+1})$ with values in $X_\eta \times M_\eta$, for each $0 \le n \le N-1$, solution of :

$$\forall z_\eta \in X_\eta, \frac{1}{k}(v_\eta^{n+1} - v_\eta^n, z_\eta) + \nu(\nabla v_\eta^{n+1}, \nabla z_\eta) - (q_\eta^{n+1}, \operatorname{div} z_\eta) = (g^{n+1}, z_\eta), \tag{4.5}$$

$$\forall q_\eta \in M_\eta, (q_\eta, \operatorname{div} v_\eta^{n+1}) = 0, \tag{4.6}$$

$$v_\eta^0 = 0 \quad \text{in} \quad \Omega, \tag{4.7}$$

where $g^{n+1}$ is the same approximation as in (3.1).

This linear problem has a unique solution that satisfies the following error estimate :

**Lemma 4.2.** *Let $\Omega$ be a convex, $g \in L^2(\Omega\times]0,T[)^2, g' \in L^2(0,T;H^{-1}(\Omega)^2)$ and $g(0) \in L^2(\Omega)^2$. Then, there exists a constant $C$ that does not depend on $\eta$ and $k$ such that*

$$\begin{aligned}
&\sup_{0\le n\le N} \| v_\eta^n - v(t^n) \|_{L^2(\Omega)} + \Big( \sum_{n=0}^{N-1} \| (v_\eta^{n+1} - v(t^{n+1})) \ - (v_\eta^n - v(t^n)) \|^2_{L^2(\Omega)} \Big)^{1/2} \\
&+ \sqrt{\nu} \Big( \sum_{n=0}^{N-1} k |v_\eta^{n+1} - v(t^{n+1})|^2_{H^1(\Omega)} \Big)^{1/2} \\
&\le C(\eta + k) \Big( \| g \|_{L^2(\Omega\times]0,T[)^2} + \| g' \|_{L^2(0,T;H^{-1}(\Omega)^2)} + \| g(0) \|_{L^2(\Omega)^2} \Big).
\end{aligned}$$

In addition, the solution $(v_\eta^{n+1}, q_\eta^{n+1})$ of (4.5)–(4.7) satisfies

**Lemma 4.3.** *In addition to the hypotheses of Lemma 4.2, suppose that $q' \in L^2(\Omega\times]0,T[)$. There exists a constant $C$ that does not depend on $\eta$ and $k$ such that*

$$\begin{aligned}
&\Big( \sum_{n=0}^{N-1} k \, \| \frac{(v_\eta^{n+1} - v(t^{n+1})) - (v_\eta^n - v(t^n))}{k} \|^2_{L^2(\Omega)} \Big)^{1/2} \quad + \quad \sqrt{\nu} \sup_{0\le n\le N} |v_\eta^n - v(t^n)|_{H^1(\Omega)} \\
&+ \sqrt{\nu} \Big( \sum_{n=0}^{N-1} |(v_\eta^{n+1} - v(t^{n+1})) - (v_\eta^n - v(t^n))|^2_{H^1(\Omega)} \Big)^{1/2} \quad \le \quad C(\eta + \sqrt{k}).
\end{aligned} \tag{4.8}$$

The parabolic duality argument (cf. [19]) consists in defining the solution $(w^n, \lambda^n)$ of the backward semi-discrete Stokes system :

$$\frac{1}{k}(w^{n+1} - w^n) + \nu \Delta w^n - \nabla \lambda^n = v_\eta^n - v(t^n) \text{ in } \Omega, \tag{4.9}$$

$$\operatorname{div} w^n = 0 \text{ in } \Omega, \tag{4.10}$$

$$w^n = 0 \text{ on } \partial\Omega, \tag{4.11}$$

$$w^{N+1} = 0 \text{ in } \Omega, \tag{4.12}$$

where $0 \leq n \leq N$.

For each $n$, knowing $w^{n+1}$, the Stokes problem (4.9)–(4.12) has a unique solution $w^n \in H_0^1(\Omega)^2$, $\lambda^n \in L_0^2(\Omega)$, (cf. [7], [18]).

The next lemma establishes basic estimates for the velocity $w^n$ of the backward semi-discrete Stokes problem (4.9)–(4.12).

**Lemma 4.4.** *Standard arguments give the uniform bounds :*

$$\begin{aligned}\sup_{0\leq n\leq N} \| w^n \|_{L^2(\Omega)} + \Big( \sum_{n=0}^{N} \| w^{n+1} - w^n \|^2_{L^2(\Omega)} \Big)^{1/2} + \sqrt{\nu}\Big( \sum_{n=0}^{N} k |w^n|^2_{H^1(\Omega)} \Big)^{1/2} \\ \leq \sqrt{\frac{3}{\nu}} S_2 \Big( \sum_{n=0}^{N} k \| v^n_\eta - v(t^n) \|^2_{L^2(\Omega)} \Big)^{1/2},\end{aligned} \tag{4.13}$$

*where $S_2$ is the constant of Poincaré's inequality, and*

$$\begin{aligned}\sup_{0\leq n\leq N} \sqrt{\nu}|w^n|_{H^1(\Omega)} + \sqrt{\nu}\Big( \sum_{n=0}^{N} |w^{n+1} - w^n|^2_{H^1(\Omega)} \Big)^{1/2} + \Big( \sum_{n=0}^{N} k \| \frac{w^{n+1} - w^n}{k} \|^2_{L^2(\Omega)} \Big)^{1/2} \\ \leq \sqrt{3} \Big( \sum_{n=0}^{N} k \| v^n_\eta - v(t^n) \|^2_{L^2(\Omega)} \Big)^{1/2}.\end{aligned} \tag{4.14}$$

*If $\Omega$ is convex,* (4.14) *implies the uniform bound*

$$\Big( \sum_{n=0}^{N} k (|w^n|^2_{H^2(\Omega)} + |\lambda^n|^2_{H^1(\Omega)}) \Big)^{1/2} \leq C \Big( \sum_{n=0}^{N} k \| v^n_\eta - v(t^n) \|^2_{L^2(\Omega)} \Big)^{1/2}, \tag{4.15}$$

*with a constant $C$ independent of $k$ and $\eta$.*

*Proof.* For the first inequality, we take the scalar product of (4.9) with $z = w^n$ and we use the incompressibility condition. This gives

$$-\frac{1}{k}(w^{n+1} - w^n, w^n) + \nu |w^n|^2_{H^1(\Omega)} = (v(t^n) - v^n_\eta, w^n)$$

Multiplying the above equation by $k$, summing it over $n$ from $i$ to $N$, and applying the Poincaré's inequality, we obtain for any $\varepsilon > 0$

$$\begin{aligned}\frac{1}{2}(\| w^i \|^2_{L^2(\Omega)} + \sum_{j=i}^{N} \| w^j - w^{j+1} \|^2_{L^2(\Omega)}) + \nu \sum_{j=i}^{N} k |w^j|^2_{H^1(\Omega)} \\ \leq \frac{1}{2} S_2 \Big( \varepsilon \sum_{j=i}^{N} k |w^j|^2_{H^1(\Omega)} + \frac{1}{\varepsilon} \sum_{j=i}^{N} k \| v(t^j) - v^j_\eta \|^2_{L^2(\Omega)} \Big),\end{aligned}$$

where $S_2$ is Poincaré's constant.
Then (4.13) follows after the suitable choice of $\varepsilon = \dfrac{\nu}{S_2}$.

Similarly, for the second inequality, we take the scalar product of (4.9) with $z = \dfrac{1}{k}(w^n - w^{n+1})$, we multiply the equation by $k$ and sum it over $n$.

Now, we assume that $\Omega$ is convex. Since (4.9)–(4.12) is a steady Stokes problem with right-hand side

$(w^{n+1}-w^n)/k + v(t^n) - v_\eta^n$, we have $w^n \in H^2(\Omega)^2, \lambda^n \in H^1(\Omega)$ (cf. [8]) and (4.14) implies also the uniform bound (4.15). □

From now on, we assume that $\Omega$ is convex. Using these inequalities, the next theorem establishes that the error satisfies an estimate of order two in $L^2(\Omega\times]0,T[)^2$.

**Theorem 4.5.** *We suppose that there exists a constant $\alpha > 0$, independent of $\eta$ and $k$, such that $k \geq \alpha\eta^2$. If $g \in L^2(\Omega\times]0,T[)^2, g' \in L^2(0,T;H^{-1}(\Omega)^2)$ and $g(0) \in L^2(\Omega)^2$, then there exists a constant $C$, independent of $\eta, k, g, g'$ and $g(0)$ such that*

$$\Big(\sum_{n=0}^{N} k||v_\eta^n - v(t^n)||^2_{L^2(\Omega)}\Big)^{1/2} \leq C(\eta^2 \ + \ k). \tag{4.16}$$

*In particular, if (3.6) holds, then*

$$\Big(\sum_{n=0}^{N} k||v_\eta^n - v(t^n)||^2_{L^2(\Omega)}\Big)^{1/2} \leq C\eta^2. \tag{4.17}$$

*Proof.* Let $e^n = v_\eta^n - v(t^n)$. On one hand, taking the scalar product of (4.9) by $e^n$, applying

$$\sum_{n=0}^{N} (a^{n+1}-a^n)b^n = a^{N+1}b^N - a^0b^0 + \sum_{n=0}^{N-1} a^{n+1}(b^n - b^{n+1}), \tag{4.18}$$

summing over $n$ and inserting $P_\eta w^{n+1}$, we obtain

$$\begin{aligned}\sum_{n=0}^{N} k \parallel e^n \parallel^2_{L^2(\Omega)} =& \sum_{n=0}^{N-1} (w^{n+1} - P_\eta w^{n+1}, e^n - e^{n+1}) \ + \sum_{n=0}^{N-1} (P_\eta w^{n+1}, e^n - e^{n+1}) \\ & -\nu \sum_{n=0}^{N} k \ (\nabla(w^n - w^{n+1}), \nabla e^n) \ - \nu \sum_{n=0}^{N} k \ (\nabla w^{n+1}, \nabla e^n) + \sum_{n=0}^{N} k \ (\lambda^n - r_\eta\lambda^n, \operatorname{div} e^n)\end{aligned} \tag{4.19}$$

because $w^{N+1} = 0, \ e^0 = 0$ and $(r_\eta\lambda^n, \operatorname{div} e^n) = (r_\eta\lambda^n, \operatorname{div} v_\eta^n) - (r_\eta\lambda^n, \operatorname{div} v(t^n)) = 0$.

On the other hand, we integrate (4.1) and (4.5) over $[t^n, t^{n+1}]$ and we take the difference between the resulting equations. This gives

$$(e^{n+1} - e^n \ , \ \varphi_\eta) = -\nu \int_{t^n}^{t^{n+1}} (\nabla(v_\eta^{n+1} - v(s)), \nabla\varphi_\eta) ds \ - \ \int_{t^n}^{t^{n+1}} (q(s) - r_\eta q(s), \operatorname{div} \varphi_\eta) ds.$$

This result is substituted into the second term of the right-hand side of (4.19) with $\varphi_\eta = P_\eta w^{n+1}$. So (4.19) becomes

$$\begin{aligned}\sum_{n=0}^{N} k \parallel e^n \parallel^2_{L^2(\Omega)} =& \sum_{n=0}^{N-1} (w^{n+1} - P_\eta w^{n+1}, e^n - e^{n+1}) \ + \sum_{n=0}^{N-1} \nu \int_{t^n}^{t^{n+1}} (\nabla(v_\eta^{n+1} - v(s)), \nabla P_\eta w^{n+1}) ds \\ & + \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} (q(s) - r_\eta q(s), \operatorname{div} P_\eta w^{n+1}) ds \ - \nu \sum_{n=0}^{N} k \ (\nabla(w^n - w^{n+1}), \nabla e^n) \\ & -\nu \sum_{n=0}^{N} k \ (\nabla(w^{n+1} - P_\eta w^{n+1}), \nabla e^n) \ - \nu \sum_{n=0}^{N} k(\nabla P_\eta w^{n+1}, \nabla e^n) + \sum_{n=0}^{N} k \ (\lambda^n - r_\eta\lambda^n, \operatorname{div} e^n).\end{aligned} \tag{4.20}$$

Inserting $\pm\nabla v(t^{n+1})$ in the second term of the right hand side and using the formula

$$\int_{t^n}^{t^{n+1}} \int_{s}^{t^{n+1}} \nabla v'(\tau) d\tau ds = \int_{t^n}^{t^{n+1}} (\tau - t^n) \nabla v'(\tau) d\tau,$$

this term becomes

$$\sum_{n=0}^{N-1} \nu \int_{t^n}^{t^{n+1}} (\nabla e^{n+1}, \nabla P_\eta w^{n+1}) ds + \sum_{n=0}^{N-1} \nu \int_{t^n}^{t^{n+1}} (\tau - t^n)(\nabla v'(\tau), \nabla P_\eta w^{n+1}) d\tau.$$

The sixth term can be written as follows :

$$\begin{aligned} &\sum_{n=0}^{N} k(\nabla P_\eta w^{n+1}, \nabla(e^n - e^{n+1})) + \sum_{n=0}^{N} k(\nabla P_\eta w^{n+1}, \nabla e^{n+1}) \\ &= \sum_{n=0}^{N} k(\nabla P_\eta (w^{n+1} - w^n), \nabla e^n) + \sum_{n=0}^{N} k(\nabla P_\eta w^{n+1}, \nabla e^{n+1}). \end{aligned}$$

Replacing them in (4.20) and using (4.18), we obtain

$$\begin{aligned} \sum_{n=0}^{N} k \parallel e^n \parallel^2_{L^2(\Omega)} \;=\; & \sum_{n=0}^{N-1} (w^{n+1} - P_\eta w^{n+1}, e^n - e^{n+1}) \;+ \sum_{n=0}^{N-1} \nu \int_{t^n}^{t^{n+1}} (\tau - t^n)(\nabla v'(\tau), \nabla P_\eta w^{n+1}) d\tau \\ & -\nu \sum_{n=0}^{N} k(\nabla(P_\eta w^n - w^n), \nabla e^n) + \sum_{n=0}^{N} k(\lambda^n - r_\eta \lambda^n, \operatorname{div} e^n) \\ & + \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} (q(s) - r_\eta q(s), \operatorname{div}(P_\eta w^{n+1} - w^{n+1})) ds. \end{aligned} \tag{4.21}$$

Denote the terms in the right-hand side of (4.21) by $(W_{RH})_j, j = 1, ..., 5$. Using the approximation properties of $P_\eta$, applying (4.13) and Lemma 4.2, the first and second terms can be bounded as follows :

$$\begin{aligned} |(W_{RH})_1| &\leq C \frac{\eta^2}{\sqrt{k}} \Big( \sum_{n=0}^{N-1} \parallel e^n - e^{n+1} \parallel^2_{L^2(\Omega)} \Big)^{1/2} \Big( \sum_{n=0}^{N-1} k \parallel w^{n+1} \parallel^2_{H^2(\Omega)} \Big)^{1/2} \\ &\leq \frac{C\eta^2}{\sqrt{k}} (\eta + k) \Big( \sum_{n=0}^{N-1} k \parallel e^n \parallel^2_{L^2(\Omega)} \Big)^{1/2}, \end{aligned}$$

$$\begin{aligned} |(W_{RH})_2| &\leq \frac{k}{\sqrt{3}} \parallel v' \parallel_{L^2(0,T;\, H^1(\Omega)^2)} \Big( \sum_{n=0}^{N-1} k \parallel \nabla P_\eta w^{n+1} \parallel^2_{L^2(\Omega)} \Big)^{1/2} \\ &\leq \frac{C}{\sqrt{3}} k \parallel v' \parallel_{L^2(0,T;H^1(\Omega)^2)} \Big( \sum_{n=0}^{N} k \parallel e^n \parallel^2_{L^2(\Omega)} \Big)^{1/2}. \end{aligned}$$

Owing to Lemma 4.2 and (4.15), the third and fourth terms can be bounded by :

$$|(W_{RH})_3| \leq C\eta(\eta + k) \Big( \sum_{n=0}^{N} k \parallel e^n \parallel^2_{L^2(\Omega)} \Big)^{1/2},$$

$$\begin{aligned} |(W_{RH})_4| &\leq \eta \Big( \sum_{n=0}^{N} k |\lambda^n|^2_{H^1(\Omega)} \Big)^{1/2} \Big( \sum_{n=0}^{N} k |e^n|^2_{H^1(\Omega)} \Big)^{1/2} \\ &\leq C\eta(\eta + k) \Big( \sum_{n=0}^{N} k \parallel e^n \parallel^2_{L^2(\Omega)} \Big)^{1/2}. \end{aligned}$$

Finally, the last term is bounded by :

$$\begin{aligned} |(W_{RH})_5| &\leq C\eta \Big( \sum_{n=0}^{N-1} k \parallel q - r_\eta q \parallel^2_{L^2(\Omega)} \Big)^{1/2} \Big( \sum_{n=0}^{N-1} k \parallel w^{n+1} \parallel^2_{H^2(\Omega)} \Big)^{1/2} \\ &\leq C\eta^2 \parallel q \parallel_{L^2(0,T;H^1(\Omega))} \Big( \sum_{n=0}^{N-1} k \parallel e^n \parallel^2_{L^2(\Omega)} \Big)^{1/2}. \end{aligned}$$

Substituting these inequalities into (4.21) we obtain (4.16).

If (3.6) holds, then (4.16) implies (4.17). $\square$

Now, we split $u_\eta^n - u(t^n)$ into a linear contribution, $v_\eta^n - u(t^n)$ and a non-linear one $u_\eta^n - v_\eta^n$. Here $v_\eta^{n+1}$ is the solution of the Stokes problem (4.5)–(4.7) with $g = f - u.\nabla u$. Therefore, $v = u$ and $v_\eta^{n+1}$ solves the discrete problem

$$\frac{1}{k}(v_\eta^{n+1} - v_\eta^n, w_\eta) \ + \ \nu(\nabla v_\eta^{n+1}, \nabla w_\eta) - (q_\eta^{n+1}, \operatorname{div} w_\eta) = \frac{1}{k}\int_{t^n}^{t^{n+1}} (f(s) - u(s).\nabla u(s), w_\eta) ds \tag{4.22}$$

with (4.6)–(4.7).

On one hand, by assumption (1.8), we have $f - u.\nabla u \in L^2(\Omega\times]0,T[)^2$. Therefore, Theorem 4.5 gives

$$\Big(\sum_{n=0}^{N} k||v_\eta^n - u(t^n)||^2_{L^2(\Omega)}\Big)^{1/2} \leq C(f,u,p,\nu,T)(\eta^2 \ + \ k), \tag{4.23}$$

with another constant $C(f,u,p,\nu,T)$ that does not depend on $\eta^2$ nor on $k$.

Furthermore, if $p'$ belongs to $L^2(\Omega\times]0,T[)$, Lemma 4.3 implies that

$$\sup_{0\leq n\leq N} |v_\eta^n - u(t^n)|_{H^1(\Omega)} \leq C(\eta + \sqrt{k}).$$

On the other hand, we prove the following "superconvergence" result for the non-linear part.

**Theorem 4.6.** *Assume that $f \in L^\infty(0,T;L^2(\Omega)^2), p' \in L^2(\Omega\times]0,T[), u \in \mathcal{C}^0(0,T;W^{1,4}(\Omega)^2)$ and $u' \in L^2(0,T;H^1(\Omega)^2)$, then there exists a constant $C$ that does not depend on $\eta$ and $k$, such that*

$$\begin{aligned} \sup_{0\leq n\leq N} \parallel v_\eta^n - u_\eta^n \parallel_{L^2(\Omega)} &+ \Big(\sum_{n=0}^{N-1} \parallel (v_\eta^{n+1} - u_\eta^{n+1}) - (v_\eta^n - u_\eta^n) \parallel^2_{L^2(\Omega)}\Big)^{1/2} \\ &+ \sqrt{\nu}\Big(\sum_{n=0}^{N-1} k|v_\eta^{n+1} - u_\eta^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2} \leq C(\eta^2 \ + \ k). \end{aligned} \tag{4.24}$$

*Proof.* By taking the difference between (4.22) and (1.18), we obtain :

$$\begin{aligned} &\forall \varphi_\eta \in V_\eta, \ \ \frac{1}{k}((v_\eta^{n+1} - u_\eta^{n+1}) - (v_\eta^n - u_\eta^n), \varphi_\eta) + \nu(\nabla(v_\eta^{n+1} - u_\eta^{n+1}), \nabla\varphi_\eta) \\ &= \frac{1}{k}\int_{t^n}^{t^{n+1}} \Big[(u_\eta^{n+1}\cdot\nabla u_\eta^{n+1} - u(s)\cdot\nabla u(s), \varphi_\eta) + \frac{1}{2}(\operatorname{div} u_\eta^{n+1}, u_\eta^{n+1}\cdot\varphi_\eta)\Big] ds. \end{aligned} \tag{4.25}$$

We split $u(s)\cdot\nabla u(s) - u_\eta^{n+1}\cdot\nabla u_\eta^{n+1}$ as follows :

$$\begin{aligned} &u(s)\cdot\nabla u(s) - u_\eta^{n+1}\cdot\nabla u_\eta^{n+1} = (u(s) - u(t^{n+1}))\cdot\nabla u(s) + u(t^{n+1})\cdot\nabla(u(s) - u(t^{n+1})) \\ &\quad -(u_\eta^{n+1} - v_\eta^{n+1})\cdot\nabla u_\eta^{n+1} - \ v_\eta^{n+1}\cdot\nabla(u_\eta^{n+1} - v_\eta^{n+1}) - \ (v_\eta^{n+1} - u(t^{n+1}))\cdot\nabla(v_\eta^{n+1} - u(t^{n+1})) \\ &\quad - \ (v_\eta^{n+1} - u(t^{n+1}))\cdot\nabla u(t^{n+1}) \ - \ u(t^{n+1})\cdot\nabla(v_\eta^{n+1} - u(t^{n+1})), \end{aligned}$$

and we split similarly the divergence term. To simplify, we denote by $b(u;v,w)$ the sum of these two non-linear terms;

$$b(u;v,w) = (u\cdot\nabla v, w) + \frac{1}{2}(\operatorname{div} u, v\cdot w).$$

Now, we multiply (4.25) by $k$, choose $\varphi_\eta = \varphi_\eta^{n+1} = v_\eta^{n+1} - u_\eta^{n+1}$ which belongs to $V_\eta$, and sum it over $n = 0, ..., m-1$. We obtain:

$$\begin{aligned}
&\frac{1}{2} \parallel \varphi_\eta^m \parallel^2_{L^2(\Omega)} + \frac{1}{2} \sum_{n=0}^{m-1} \parallel \varphi_\eta^{n+1} - \varphi_\eta^n \parallel^2_{L^2(\Omega)} + \nu \sum_{n=0}^{m-1} k |\varphi_\eta^{n+1}|^2_{H^1(\Omega)} \\
&= \sum_{n=0}^{m-1} \int_{t^n}^{t^{n+1}} ((u(t^{n+1}) - u(s)) \cdot \nabla u(s), \varphi_\eta^{n+1}) ds + \sum_{n=0}^{m-1} \int_{t^n}^{t^{n+1}} (u(t^{n+1}) \cdot \nabla (u(t^{n+1}) - u(s)), \varphi_\eta^{n+1}) ds \\
&\quad - \sum_{n=0}^{m-1} k \; b(\varphi_\eta^{n+1}; u_\eta^{n+1}, \varphi_\eta^{n+1}) + \sum_{n=0}^{m-1} k \Big\{ b(v_\eta^{n+1} - u(t^{n+1}); v_\eta^{n+1} - u(t^{n+1}), \varphi_\eta^{n+1}) \\
&\quad + b(v_\eta^{n+1} - u(t^{n+1}); u(t^{n+1}), \varphi_\eta^{n+1}) \Big\} + \sum_{n=0}^{m-1} k \; b(u(t^{n+1}); v_\eta^{n+1} - u(t^{n+1}), \varphi_\eta^{n+1}).
\end{aligned} \tag{4.26}$$

We note $(U_{RH})_i, i = 1, ..., 6$, the terms in the right-hand side of (4.26) and set

$$C_0 = \parallel u \parallel_{L^\infty(0,T;L^4(\Omega)^2)} \text{ and } \widehat{C} = \parallel u' \parallel_{L^2(0,T;L^4(\Omega)^2)} .$$

For the first two terms, since div $u = 0$, we can write

$$((u(t^{n+1}) - u(s)) \cdot \nabla u(s), \varphi_\eta^{n+1}) = -((u(t^{n+1}) - u(s)) \cdot \nabla \varphi_\eta^{n+1}, u(s)),$$

$$(u(t^{n+1}) \cdot \nabla (u(t^{n+1}) - u(s)), \varphi_\eta^{n+1}) = -(u(t^{n+1}) \cdot \nabla \varphi_\eta^{n+1}, u(t^{n+1}) - u(s)).$$

Therefore, for any $\varepsilon_0 > 0$,

$$|(U_{RH})_1| \leq \frac{\widehat{C}}{2\sqrt{3}} \Big\{ \frac{C_0^2}{\varepsilon_0} k^2 + \varepsilon_0 \sum_{n=0}^{m-1} k |\varphi_\eta^{n+1}|^2_{H^1(\Omega)} \Big\},$$

with the same bound for $(U_{RH})_2$.

For the third term, we set $C_1 = \sup_{0 \leq n \leq N} |u_\eta^n|_{H^1(\Omega)}$. The two parts are treated similarly and we obtain, for any $\varepsilon_1 > 0$,

$$|(U_{RH})_3| \leq 2^{1/2} C_1 \Big\{ \varepsilon_1 \sum_{n=0}^{m-1} k |\varphi_\eta^{n+1}|^2_{H^1(\Omega)} + \frac{1}{\varepsilon_1} \sum_{n=0}^{m-1} k \parallel \varphi_\eta^{n+1} \parallel^2_{L^2(\Omega)} \Big\}.$$

In order to bound the two last terms, we use the well-known formula

$$b(u; v, w) = \frac{1}{2} \Big[ \int_\Omega (u \cdot \nabla v) \cdot w - \int_\Omega (u \cdot \nabla w) \cdot v \Big].$$

The fourth term is split into two parts that we treat successively : Using Lemma 4.2 and Lemma 4.3, for any $\varepsilon_2 > 0$, we bound the first part as follows :

$$\begin{aligned}
|(U_{RH})_{4,1}| &= \Big| \frac{1}{2} \sum_{n=0}^{m-1} k \; ((v_\eta^{n+1} - u(t^{n+1})) \cdot \nabla (v_\eta^{n+1} - u(t^{n+1})), \varphi_\eta^{n+1}) \Big| \\
&\leq \frac{C S_4^2}{2} (\eta + k) \Big( \sum_{n=0}^{m-1} k |\varphi_\eta^{n+1}|^2_{H^1(\Omega)} \Big)^{1/2} \Big( \sum_{n=0}^{m-1} k |v_\eta^{n+1} - u(t^{n+1})|^2_{H^1(\Omega)} \Big)^{1/2} \\
&\leq \frac{S_4^2}{2} \Big\{ \frac{C^2}{\varepsilon_2} (\eta^4 + k^3 + \eta^2 k) + \varepsilon_2 \sum_{n=0}^{m-1} k |\varphi_\eta^{n+1}|^2_{H^1(\Omega)} \Big\},
\end{aligned}$$

and the second part is bounded exactly as the first part. For any $\varepsilon_3 > 0$,

$$
\begin{aligned}
|(U_{RH})_{4,2}| &= \Big|\frac{1}{2}\sum_{n=0}^{m-1} k(v_\eta^{n+1} - u(t^{n+1})\cdot\nabla\varphi_\eta^{n+1}, v_\eta^{n+1} - u(t^{n+1}))\Big| \\
&\le \frac{S_4^2}{2}\sup_n |v_\eta^{n+1} - u(t^{n+1})|_{H^1(\Omega)}\Big(\sum_{n=0}^{m-1} k|\varphi_\eta^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2}\Big(\sum_{n=0}^{m-1} k|v_\eta^{n+1} - u(t^{n+1})|^2_{H^1(\Omega)}\Big)^{1/2} \\
&\le \frac{S_4^2}{2}\Big\{\frac{C^2}{\varepsilon_3}(\eta^4 + k^3 + \eta^2 k) + \varepsilon_3\sum_{n=0}^{m-1} k|\varphi_\eta^{n+1}|^2_{H^1(\Omega)}\Big\}.
\end{aligned}
$$

The fifth term is bounded as the fourth term. Setting $C_2 = \| u \|_{L^\infty(0,T;W^{1,4}(\Omega)^2)}$, for any $\varepsilon_4 > 0$, the first part is bounded as follows :

$$
\begin{aligned}
|(U_{RH})_{5,1}| &= \Big|\frac{1}{2}\sum_{n=0}^{m-1} k((v_\eta^{n+1} - u(t^{n+1}))\cdot\nabla u(t^{n+1}), \varphi_\eta^{n+1})\Big| \\
&\le S_4 C_2(\eta^2 + k)(\sum_{n=0}^{m-1} k|\varphi_\eta^{n+1}|^2_{H^1(\Omega)})^{1/2} \\
&\le \frac{S_4 C_2}{2}\Big\{\frac{1}{\varepsilon_4}(\eta^4 + k^2) + \varepsilon_4\sum_{n=0}^{m-1} k|\varphi_\eta^{n+1}|^2_{H^1(\Omega)}\Big\},
\end{aligned}
$$

and for any $\varepsilon_5 > 0$, the second part is bounded as follows :

$$
\begin{aligned}
|(U_{RH})_{5,2}| &= \Big|\frac{1}{2}\sum_{n=0}^{m-1} k(v_\eta^{n+1} - u(t^{n+1})\cdot\nabla\varphi_\eta^{n+1}, u(t^{n+1}))\Big| \\
&\le \frac{C_2}{4}\Big\{\frac{1}{\varepsilon_5}(k^2 + \eta^4) + \varepsilon_5\sum_{n=0}^{m-1} k|\varphi_\eta^{n+1}|^2_{H^1(\Omega)}\Big\}.
\end{aligned}
$$

Finally, the last term is bounded by applying Green's formula : For any $\varepsilon_6 > 0$,

$$
\begin{aligned}
|(U_{RH})_6| &= \Big|\sum_{n=0}^{m-1} k\ (u(t^{n+1})\cdot\nabla\varphi_\eta^{n+1}, v_\eta^{n+1} - u(t^{n+1}))\Big| \\
&\le \frac{C_2}{2}\Big\{\frac{1}{\varepsilon_6}\sum_{n=0}^{m-1} k \parallel v_\eta^{n+1} - u(t^{n+1}) \parallel^2_{L^2(\Omega)} + \varepsilon_6\sum_{n=0}^{m-1} k|\varphi_\eta^{n+1}|^2_{H^1(\Omega)}\Big\} \\
&\le \frac{C_2}{2}\Big\{\frac{1}{\varepsilon_6}(\eta^4 + k^2) + \varepsilon_6\sum_{n=0}^{m-1} k|\varphi_\eta^{n+1}|^2_{H^1(\Omega)}\Big\}.
\end{aligned}
$$

Then (4.26) becomes :

$$
\begin{aligned}
&\frac{1}{2}\parallel v_\eta^m - u_\eta^m \parallel^2_{L^2(\Omega)} + \frac{1}{2}\sum_{n=0}^{m-1}\parallel (v_\eta^{n+1} - u_\eta^{n+1}) - (v_\eta^n - u_\eta^n) \parallel^2_{L^2(\Omega)} + \nu\sum_{n=0}^{m-1} k|v_\eta^{n+1} - u_\eta^{n+1}|^2_{H^1(\Omega)} \\
&\qquad \le A + B + D,
\end{aligned}
$$

where $A \le C(\eta^4 + k^2)$, $B = \gamma_1\sum_{n=0}^{m-1} k \parallel v_\eta^{n+1} - u_\eta^{n+1} \parallel^2_{L^2(\Omega)}$, $D = \gamma_2\sum_{n=0}^{m-1} k|v_\eta^{n+1} - u_\eta^{n+1}|^2_{H^1(\Omega)}$,

$$\gamma_1 = \gamma(C, \varepsilon_1, \varepsilon_5), \quad \gamma_2 = \gamma(S_4, C_i, \varepsilon_j, i = 1, 2, j = 0, ..., 6).$$

Then, after a suitable choice of $\varepsilon_i$ and applying Gronwall's lemma, the equation becomes :

$$
\begin{aligned}
&\parallel v_\eta^m - u_\eta^m \parallel^2_{L^2(\Omega)} + \sum_{n=0}^{m-1}\parallel (v_\eta^{n+1} - u_\eta^{n+1}) - (v_\eta^n - u_\eta^n) \parallel^2_{L^2(\Omega)} + \nu\sum_{n=0}^{m-1} k|v_\eta^{n+1} - u_\eta^{n+1}|^2_{H^1(\Omega)} \\
&\qquad \le e^{CT}(\eta^4 + k^2).
\end{aligned}
$$

Then the result follows from this inequality. $\square$

Combining Theorems 4.5 and 4.6, we obtain :

**Corollary 4.7.** *Under the assumptions of Theorem* 4.6, *there exists a constant* $C$, *that does not depend on* $\eta$ *and* $k$, *such that*

$$\Big(\sum_{n=0}^{N} k \parallel u(t^n) - u_\eta^n \parallel_{L^2(\Omega)}^2 \Big)^{1/2} \leq C(\eta^2 \ + \ k). \tag{4.27}$$

## 5. An estimate for the pressure

The results of the preceding section allow one to establish an error estimate for the pressure. We start with a general bound.

**Lemma 5.1.** *Under the assumptions of Corollary* 3.2, *suppose that* $p' \in L^2(\Omega\times]0,T[)$. *Let* $(u(t^{n+1}), p(t^{n+1}))$ *and* $(u_\eta^{n+1}, p_\eta^{n+1})$ *be the respective solutions of* (1.1)–(1.4) *and* (1.18)–(1.19). *We have*

$$\Big(\sum_{n=0}^{N-1} k \parallel p_\eta^{n+1} - r_\eta p(t^{n+1}) \parallel_{L^2(\Omega)}^2 \Big)^{1/2} \leq \frac{1}{\beta^\star}\Big\{ S_2 \Big(\sum_{n=0}^{N-1} k \parallel \frac{(u_\eta^{n+1} - u(t^{n+1})) - (u_\eta^n - u(t^n))}{k} \parallel_{L^2(\Omega)}^2 \Big)^{1/2}$$
$$+C_1(\eta + k) + C_2 k \parallel p' \parallel_{L^2(\Omega\times]0,T[)} + C_3 \eta \parallel p \parallel_{L^2(0,T;H^1(\Omega))} \Big\}, \tag{5.1}$$

*where* $\beta^\star$ *is the constant of the inf-sup condition* (1.10) *and the coefficients* $C_i, 1 \leq i \leq 3$, *are independent of* $\eta$ *and* $k$.

*Proof.* Integrate (1.1) over $[t^n, t^{n+1}]$, substract (1.18), both multiplied by a test function $w_\eta^{n+1}$, insert $r_\eta p(s)$ and $r_\eta p(t^{n+1})$ and sum the resulting equation over $n$ from 0 to $N-1$. This gives

$$\sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} (p_\eta^{n+1} - r_\eta p(t^{n+1}), \operatorname{div} w_\eta^{n+1}) ds$$
$$= \sum_{n=0}^{N-1} \Big\{ (\varphi^{n+1}(t^{n+1}) - \varphi^n(t^n), w_\eta^{n+1}) + \nu \int_{t^n}^{t^{n+1}} (\nabla \varphi^{n+1}(s), \nabla w_\eta^{n+1}) ds$$
$$+ \Big( \int_{t^n}^{t^{n+1}} (\varphi^{n+1}(s) \cdot \nabla u_\eta^{n+1}, w_\eta^{n+1}) ds + \frac{1}{2} \int_{t^n}^{t^{n+1}} (\operatorname{div} \varphi^{n+1}(s), u_\eta^{n+1} \cdot w_\eta^{n+1}) ds \Big)$$
$$+ \int_{t^n}^{t^{n+1}} (u(s) \cdot \nabla \varphi^{n+1}(s), w_\eta^{n+1}) ds + \int_{t^n}^{t^{n+1}} (r_\eta p(s) - r_\eta p(t^{n+1}), \operatorname{div} w_\eta^{n+1}) ds$$
$$+ \int_{t^n}^{t^{n+1}} (p(s) - r_\eta p(s), \operatorname{div} w_\eta^{n+1}) ds \Big\}, \tag{5.2}$$

where $\varphi^i(\tau) = u_\eta^i - u(\tau)$.

Owing to the inf-sup condition (1.10), there exists a function $w_\eta \in V_\eta^\perp$ such that

$$(\operatorname{div} w_\eta, p_\eta^{n+1} - r_\eta p(t^{n+1})) = \parallel p_\eta^{n+1} - r_\eta p(t^{n+1}) \parallel_{L^2(\Omega)}^2 \ , \ |w_\eta|_{H^1(\Omega)} \leq \frac{1}{\beta^\star} \parallel p_\eta^{n+1} - r_\eta p(t^{n+1}) \parallel_{L^2(\Omega)} .$$

Let $(P_{RH})_i, i = 1, ..., 6$, denote the terms of the right-hand side of (5.2).

We deduce by standard arguments and by using the estimate (3.2) :

$$|(P_{RH})_1| \le S_2\Big(\sum_{n=0}^{N-1} k \parallel \frac{(u_\eta^{n+1}-u(t^{n+1}))-(u_\eta^n-u(t^n))}{k} \parallel^2_{L^2(\Omega)}\Big)^{1/2}\Big(\sum_{n=0}^{N-1} k|w_\eta^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2},$$

$$\begin{aligned}|(P_{RH})_2| &\le \nu\Big(\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}} |u_\eta^{n+1}-u(s)|^2_{H^1(\Omega)}ds\Big)^{1/2}\Big(\sum_{n=0}^{N-1} k|w_\eta^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2}\\ &\le C_1\eta\Big(\sum_{n=0}^{N-1} k|w_\eta^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2},\end{aligned}$$

$$\begin{aligned}|(P_{RH})_3| &\le S_4^2\sup_n \parallel \nabla u_\eta^{n+1}\parallel_{L^2(\Omega)}\Big(\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}} |\varphi^{n+1}(s)|^2_{H^1(\Omega)}ds\Big)^{1/2}\Big(\sum_{n=0}^{N-1} k|w_\eta^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2}\\ &\le C_2(\eta+k)\Big(\sum_{n=0}^{N-1} k|w_\eta^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2}.\end{aligned}$$

As far as $(P_{RH})_4$ is concerned, since

$$(P_{RH})_4 = -\int_{t^n}^{t^{n+1}} (u(s)\cdot\nabla w_\eta^{n+1}, \varphi^{n+1}(s))ds,$$

we have

$$\begin{aligned}|(P_{RH})_4| &\le S_4^2 \parallel u \parallel_{L^\infty(0,T;L^4(\Omega)^2)}\Big(\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}} |\varphi^{n+1}(s)|^2_{H^1(\Omega)}\Big)^{1/2}\Big(\sum_{n=0}^{N-1} k|w_\eta^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2}\\ &\le C_3(\eta+k)\Big(\sum_{n=0}^{N-1} k|w_\eta^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2}.\end{aligned}$$

$$|(P_{RH})_5| \le \frac{Ck}{\sqrt{3}} \parallel p' \parallel_{L^2(\Omega\times]0,T[)}\Big(\sum_{n=0}^{N-1} k|w_\eta^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2},$$

and

$$|(P_{RH})_6| \le C\eta \parallel p \parallel_{L^2(0,T;H^1(\Omega))}\Big(\sum_{n=0}^{N-1} k|w_\eta^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2}.$$

Then (5.1) follows easily by substituting these inequalities into (5.2). □

We have to estimate $\dfrac{(u_\eta^{n+1}-u(t^{n+1}))-(u_\eta^n-u(t^n))}{k}$ in $L^2(\Omega\times]0,T[)^2$. This estimate is proven assuming the triangulation satisfies a milder regularity property than uniform regularity (1.9): in addition to this property, there exists a constante $\tilde\tau$ that does not depend on $\eta$ or $k$ such that

$$\rho_{\min} \ge \tilde\tau\eta^5, \quad \text{where } \rho_{\min} = \inf_{\kappa\in\mathcal{T}_\eta}\rho_\kappa. \tag{5.3}$$

More precisely, this assumption is used in proving that $u_\eta^n$ is bounded in $L^\infty(0,T;W^{1,5/2}(\Omega)^2)$.

**Lemma 5.2.** *Under the assumptions of Theorem* 4.6 *and if* $\mathcal{T}_\eta$ *satisfies* (5.3), *there exists a constant* $C$ *that depends neither on* $\eta$ *nor on* $k$, *such that*

$$\sup_n |u_\eta^n|_{W^{1,5/2}(\Omega)} \le C. \tag{5.4}$$

*Proof.* Lets us sketch the proof. We write

$$|u_\eta^n|_{W^{1,5/2}(\Omega)} \leq |u_\eta^n - v_\eta^n|_{W^{1,5/2}(\Omega)} + |v_\eta^n - P_\eta u(t^n)|_{W^{1,5/2}(\Omega)} + |P_\eta u(t^n) - u(t^n)|_{W^{1,5/2}(\Omega)}$$
$$+|u(t^n)|_{W^{1,5/2}(\Omega)}.$$

To evaluate the first and second terms in the right-hand side of the above inequality, we consider the reference element $\widehat{\kappa}$, where all norms are equivalent, revert to the element $\kappa$, sum over all $\kappa \in \mathcal{T}_\eta$, apply Jensen's inequality and the regularity of $\mathcal{T}_\eta$, we obtain then an inverse inequality. Then, we obtain

$$|u_\eta^n|_{W^{1,5/2}(\Omega)} \leq C_1 + C_2 |u(t^n)|_{W^{1,5/2}(\Omega)}.$$

As we have $\sup\limits_n |u(t^n)|_{W^{1,5/2}(\Omega)} \leq C$, the result follows easily. □

**Lemma 5.3.** *Under the assumptions of Theorem* 4.6 *and Lemma* 5.2, *there exists a constant* $C = C(u, u', p', u_\eta, \nu)$ *that does not depend on* $\eta$ *or on* $k$, *such that*

$$\Big(\sum_{n=0}^{N-1} k \parallel \frac{(u_\eta^{n+1} - u(t^{n+1})) - (u_\eta^n - u(t^n))}{k} \parallel_{L^2(\Omega)}^2 \Big)^{1/2} + \sqrt{\nu} \sup_{0\leq n\leq N} |u_\eta^n - u(t^n)|_{H^1(\Omega)}$$
$$+\sqrt{\nu}\Big(\sum_{n=0}^{N-1} |(u_\eta^{n+1} - u(t^{n+1})) - (u_\eta^n - u(t^n))|_{H^1(\Omega)}^2\Big)^{1/2} \leq C(\eta + \sqrt{k}). \tag{5.5}$$

*Proof.* The proof is similar to that of Lemma 5.1. But here we also insert $S_\eta u(s)$, where $S_\eta$ is defined by

$\forall (u,p) \in V \times L_0^2(\Omega), S_\eta(u) \in V_\eta,$

$$\forall v_\eta \in V_\eta, \;\; \nu(\nabla(S_\eta(u) - u), \nabla v_\eta) \; = \; -(p, \operatorname{div} v_\eta), \tag{5.6}$$

and we take $e_\eta^n = u_\eta^n - S_\eta u(t^n)$. Then we obtain

$$\sum_{n=0}^{m-1} k \parallel \frac{e_\eta^{n+1} - e_\eta^n}{k} \parallel_{L^2(\Omega)}^2 + \frac{\nu}{2}(\parallel \nabla e_\eta^m \parallel_{L^2(\Omega)}^2 - \parallel \nabla e_\eta^0 \parallel_{L^2(\Omega)}^2 + \sum_{n=0}^{m-1} \parallel \nabla(e_\eta^{n+1} - e_\eta^n) \parallel_{L^2(\Omega)}^2)$$
$$= \sum_{n=0}^{m-1} \int_{t^n}^{t^{n+1}} (u'(s) - S_\eta u'(s), \frac{e_\eta^{n+1} - e_\eta^n}{k}) ds - \nu \sum_{n=0}^{m-1} \int_{t^n}^{t^{n+1}} (\nabla S_\eta u'(s), \nabla(\frac{e_\eta^{n+1} - e_\eta^n}{k}))(s - t^n) ds$$
$$- \sum_{n=0}^{m-1} \int_{t^n}^{t^{n+1}} \Big((u_\eta^{n+1} \cdot \nabla u_\eta^{n+1} - u(s) \cdot \nabla u(s) + \frac{1}{2}(\operatorname{div} u_\eta^{n+1} \nabla u_\eta^{n+1} - \operatorname{div} u(s) \nabla u(s)), \frac{e_\eta^{n+1} - e_\eta^n}{k})\Big) ds. \tag{5.7}$$

Let us estimate the three terms $(V_{RH})_i, i = 1, ..., 3$, in the right-hand side of (5.7).

Using the fact that

$$\parallel S_\eta u - u \parallel_{L^2(\Omega)} \leq C\eta(|S_\eta u - u|_{H^1(\Omega)} + \parallel r_\eta p - p \parallel_{L^2(\Omega)}),$$

and

$$|S_\eta u - u|_{H^1(\Omega)} \leq 2|P_\eta u - u|_{H^1(\Omega)} + \frac{\sqrt{3}}{\nu} \parallel r_\eta p - p \parallel_{L^2(\Omega)},$$

the first term is as follows : For any $\varepsilon_0 > 0$,

$$|(V_{RH})_1| \leq \frac{C}{2\varepsilon_0}(\parallel u' \parallel_{L^2(0,T;H^1(\Omega)^2)}^2 + \parallel p' \parallel_{L^2(\Omega\times]0,T[)}^2)\eta^2 + \frac{\varepsilon_0}{2} \sum_{n=0}^{m-1} k \parallel \frac{e_\eta^{n+1} - e_\eta^n}{k} \parallel_{L^2(\Omega)}^2 .$$

Setting $\widehat{C} = \| S_\eta u' \|_{L^2(0,T;H^1(\Omega)^2)}$, the second term is bounded as follows : For any $\varepsilon_1 > 0$,

$$|(V_{RH})_2| \le \frac{\nu k}{\sqrt{3}} \sum_{n=0}^{m-1} k^{1/2} \Big| \frac{e_\eta^{n+1} - e_\eta^n}{k} \Big|_{H^1(\Omega)} \Big( \int_{t^n}^{t^{n+1}} |S_\eta u'(s)|^2_{H^1(\Omega)} ds \Big)^{1/2}$$
$$\le \frac{\nu \widehat{C}^2}{2\sqrt{3}\varepsilon_1} k + \frac{\nu \varepsilon_1}{2\sqrt{3}} \sum_{n=0}^{m-1} |e_\eta^{n+1} - e_\eta^n|^2_{H^1(\Omega)}.$$

Setting $C_0 = \| u \|_{L^\infty(\Omega\times]0,T[)^2}$, using

$$u(s) \cdot \nabla u(s) - u_\eta^{n+1} \cdot \nabla u_\eta^{n+1} - \frac{1}{2} \operatorname{div} u_\eta^{n+1} u_\eta^{n+1}$$

$$= u(s) \cdot \nabla (u(s) - u_\eta^{n+1}) + (u(s) - u_\eta^{n+1}) \cdot \nabla u_\eta^{n+1} + \frac{1}{2} \operatorname{div}(u(s) - u_\eta^{n+1}) u_\eta^{n+1},$$

the third term is bounded as follows : For any $\varepsilon_2, \varepsilon_3 > 0$,

$$|(V_{RH})_3| \le \Big(\frac{C_0 C'}{2\varepsilon_2} + \frac{C'' C'}{\varepsilon_3}\Big)(\eta^2 + k^2) + \Big(\frac{C_0 \varepsilon_2}{2} + C'' \varepsilon_3\Big) \sum_{n=0}^{m-1} k \Big\| \frac{e_\eta^{n+1} - e_\eta^n}{k} \Big\|^2_{L^2(\Omega)} .$$

Then, choosing suitably the parameters $\varepsilon_i$, the equation (5.7) becomes

$$\sum_{n=0}^{m-1} k \Big\| \frac{e_\eta^{n+1} - e_\eta^n}{k} \Big\|^2_{L^2(\Omega)} + \nu \| \nabla e_\eta^m \|^2_{L^2(\Omega)} - \nu \| \nabla e_\eta^0 \|^2_{L^2(\Omega)} + \nu \sum_{n=0}^{m-1} \| \nabla (e_\eta^{n+1} - e_\eta^n) \|^2_{L^2(\Omega)}$$

$$\le C(\eta^2 + k).$$

Finally (5.5) follows readily from this result and by applying a triangular inequality and $S_\eta$'s properties. $\square$

From these three lemmas, we easily derive an estimate of order one of the pressure.

**Theorem 5.4.** *Under the assumptions of Theorem* 4.6 *and Lemma* 5.2, *there exists a constant* $C$ *that does not depend on* $\eta$ *nor on* $k$, *such that*

$$\Big( \sum_{n=0}^{N-1} k \| p(t^{n+1}) - p_\eta^{n+1} \|^2_{L^2(\Omega)} \Big)^{1/2} \le C(\eta + \sqrt{k}). \tag{5.8}$$

*In particular, if* (3.6) *holds, then*

$$\Big( \sum_{n=0}^{N-1} k \| p(t^{n+1}) - p_\eta^{n+1} \|^2_{L^2(\Omega)} \Big)^{1/2} \le C\eta. \tag{5.9}$$

## 6. Error estimate for the solution of Step Two

We assume at this stage that we know the solution $u_H^{n+1}$ of the first step. Then at each time step, the second step (1.20)–(1.21) is a square system of linear equations in finite dimension, and if $k$ is small enough, it has a unique solution.

This solution satisfies the following error estimate.

**Theorem 6.1.** *Suppose that*

$$u \in L^\infty(0,T;H^1(\Omega)^2) \cap L^2(0,T;H^2(\Omega)^2) \cap \mathcal{C}^0(0,T;W^{1,4}(\Omega)^2),$$
$$u' \in L^2(0,T;H^1(\Omega)^2),\ p \in L^\infty(0,T;H^1(\Omega)),\ p' \in L^2(\Omega\times]0,T[),\ f \in L^2(\Omega\times]0,T[)^2,$$

*$f' \in L^2(0,T;H^{-1}(\Omega)^2)$, $f(0) \in L^2(\Omega)^2$ and $\Omega$ convex. The solution $(u_h^{n+1}, p_h^{n+1})$ of the second step satisfies the following error estimate :*

$$\sup_{0\le n\le N} \| u_h^n - u(t^n) \|_{L^2(\Omega)} + \Big( \sum_{n=0}^{N-1} \| (u_h^{n+1} - u(t^{n+1})) - (u_h^n - u(t^n)) \|^2_{L^2(\Omega)} \Big)^{1/2}$$
$$+\sqrt{\nu}\Big( \sum_{n=0}^{N-1} k|u_h^{n+1} - u(t^{n+1})|^2_{H^1(\Omega)} \Big)^{1/2} \le C(H^2 + h + k), \tag{6.1}$$

*where $C$ is a constant that does not depend on $h, H$ and $k$.*

*Proof.* On one hand, by choosing $f^{n+1}$ as in (3.1), $u_h^{n+1}$ satisfies (1.20). On the other hand, we integrate (1.5) over $[t^n, t^{n+1}]$. Then, taking the difference between the resulting equations, inserting $P_h u(t^{n+1})$ and $r_h p(s)$, choosing $v_h = v_h^{n+1} = u_h^{n+1} - P_h u(t^{n+1})$, multiplying the equation by $k$ and summing it over $n = 0, ..., m-1$, we obtain

$$\frac{1}{2}\Big( \| v_h^m \|^2_{L^2(\Omega)} + \sum_{n=0}^{m-1} \| v_h^{n+1} - v_h^n \|^2_{L^2(\Omega)} \Big) + \nu \sum_{n=0}^{m-1} k|v_h^{n+1}|^2_{H^1(\Omega)}$$
$$= \sum_{n=0}^{m-1}((u(t^{n+1}) - P_h u(t^{n+1})) - (u(t^n) - P_h u(t^n)), v_h^{n+1}) + \nu \sum_{n=0}^{m-1} \int_{t^n}^{t^{n+1}} (\nabla(u(s) - P_h u(t^{n+1})), \nabla v_h^{n+1}) ds$$
$$+ \sum_{n=0}^{m-1} \int_{t^n}^{t^{n+1}} \Big\{ (r_h p(s) - p(s), \operatorname{div} v_h^{n+1}) + (u(s)\cdot\nabla u(s) - u_H^{n+1}\cdot\nabla u_h^{n+1}, v_h^{n+1}) \Big\} ds. \tag{6.2}$$

Let us estimate the terms $(TG_{RH})_i, i = 1, ..., 4$ in the right-hand side of (6.2). The first term is bounded as follows : For any $\varepsilon_1 > 0$,

$$|(TG_{RH})_1| \le \frac{Ch^2}{2\varepsilon_1} \| u' \|^2_{L^2(0,T;H^1(\Omega)^2)} + \frac{\varepsilon_1}{2} \sum_{n=0}^{m-1} k \| v_h^{n+1} \|^2_{L^2(\Omega)} .$$

The second term is divided into two parts that we treat separately. The first part is bounded as follows : For any $\varepsilon_2 > 0$,

$$|(TG_{RH})_{2,1}| \le \frac{\nu}{2\varepsilon_2} \sum_{n=0}^{m-1} \int_{t^n}^{t^{n+1}} |u(s) - P_h u(s)|^2_{H^1(\Omega)} ds + \frac{\nu\varepsilon_2}{2} \sum_{n=0}^{m-1} k|v_h^{n+1}|^2_{H^1(\Omega)}$$
$$\le \frac{C\nu}{2\varepsilon_2} \| u \|^2_{L^2(0,T;H^2(\Omega)^2)} h^2 + \frac{\nu\varepsilon_2}{2} \sum_{n=0}^{m-1} k|v_h^{n+1}|^2_{H^1(\Omega)},$$

and the second part as follows : For any $\varepsilon_3 > 0$,

$$|(TG_{RH})_{2,2}| \le \frac{\nu k^2}{2\sqrt{3}\varepsilon_3} \| u' \|^2_{L^2(0,T;H^1(\Omega)^2)} + \frac{\nu\varepsilon_3}{2\sqrt{3}} \sum_{n=0}^{m-1} k|v_h^{n+1}|^2_{H^1(\Omega)}.$$

The third term is bounded as follows : For any $\varepsilon_4 > 0$,

$$|(TG_{RH})_3| \le \frac{1}{2\varepsilon_4} \sum_{n=0}^{m-1} \int_{t^n}^{t^{n+1}} \| r_h p(s) - p(s) \|^2_{L^2(\Omega)} ds + \frac{\varepsilon_4}{2} \sum_{n=0}^{m-1} k|v_h^{n+1}|^2_{H^1(\Omega)}$$
$$\le \frac{C}{2\varepsilon_4} \| p \|^2_{L^2(0,T;H^1(\Omega))} h^2 + \frac{\varepsilon_4}{2} \sum_{n=0}^{m-1} k|v_h^{n+1}|^2_{H^1(\Omega)}.$$

The non-linear term in the right-hand side can be written as follows :

$$u(s)\cdot\nabla u(s)-u_H^{n+1}\cdot\nabla u_h^{n+1}=(u(s)-u_H^{n+1})\cdot\nabla u(s)+u_H^{n+1}\cdot\nabla(u(s)-P_hu(t^{n+1}))$$
$$-u(t^{n+1})\cdot\nabla v_h^{n+1}-(u_H^{n+1}-u(t^{n+1}))\cdot\nabla v_h^{n+1}.$$

We study the four parts of the non-linear term $(NL)_i, i=1,...,4$, separately. The first part is treated as follows : For any $\varepsilon_5>0$,

$$\begin{aligned}
&\Big|\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}((NL)_1,v_h^{n+1})ds\Big|\\
&\le\frac{S_4}{2}\parallel u\parallel_{L^\infty(0,T;W^{1,4}(\Omega)^2)}\Big(\frac{1}{\varepsilon_5}\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\parallel u(s)-u_H^{n+1}\parallel^2_{L^2(\Omega)}ds+\varepsilon_5\sum_{n=0}^{m-1}k|v_h|^2_{H^1(\Omega)}\Big)\\
&\le\frac{S_4}{2}\parallel u\parallel_{L^\infty(0,T;W^{1,4}(\Omega)^2)}\Big(\frac{C}{\varepsilon_5}(H^4+k^2)+\varepsilon_5\sum_{n=0}^{m-1}k|v_h|^2_{H^1(\Omega)}\Big),
\end{aligned}$$

The second term bound is divided into two parts :

$$\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}((NL)_2,v_h^{n+1})ds=\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}((NL)_{2,1}+(NL)_{2,2},v_h^{n+1})ds$$

with for any $\varepsilon_6,\varepsilon_7>0$,

$$\begin{aligned}
\Big|\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}((NL)_{2,1},v_h^{n+1})ds\Big|&=\Big|\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}(u_H^{n+1}\cdot\nabla(u(s)-P_hu(s)),v_h^{n+1})ds\Big|\\
&\le\frac{S_4^2(\sup_n|u_H^n|_{H^1(\Omega)})}{2}\Big\{\frac{C}{\varepsilon_6}h^2+\varepsilon_6\sum_{n=0}^{m-1}k|v_h^{n+1}|^2_{H^1(\Omega)}\Big\},
\end{aligned}$$

and

$$\begin{aligned}
\Big|\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}((NL)_{2,2},v_h^{n+1})ds\Big|&=\Big|\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}(u_H^{n+1}\cdot\nabla P_h(u(s)-u(t^{n+1})),v_h^{n+1})ds\Big|\\
&\le\frac{S_4^2(\sup_n|u_H^n|_{H^1(\Omega)})}{2\sqrt{3}}\Big\{\frac{\parallel u'\parallel^2_{L^2(0,T;H^1(\Omega)^2)}}{\varepsilon_7}k^2+\varepsilon_7\sum_{n=0}^{m-1}k|v_h^{n+1}|^2_{H^1(\Omega)}\Big\}.
\end{aligned}$$

The third term vanishes :

$$|\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}((NL)_3,v_h^{n+1})ds|=0.$$

Finally, the last part is bounded as follows : For any $\varepsilon_8>0$,

$$\begin{aligned}
\Big|\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}((NL)_4,v_h^{n+1})ds\Big|&=\Big|\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}((u_H^{n+1}-u(t^{n+1}))\cdot\nabla v_h^{n+1},v_h^{n+1})ds\Big|\\
&\le\frac{S_4C}{2^{3/4}}\Big\{\varepsilon_8\sum_{n=0}^{m-1}k|v_h^{n+1}|^2_{H^1(\Omega)}+\frac{1}{2\varepsilon_8}\sum_{n=0}^{m-1}k\Big(\delta|v_h^{n+1}|^2_{H^1(\Omega)}+\frac{1}{\delta}\parallel v_h^{n+1}\parallel^2_{L^2(\Omega)}\Big)\Big\}.
\end{aligned}$$

Then, collecting these inequalities and choosing suitably the parameters $\varepsilon_i$ and $\delta$, and applying Gronwall's Lemma, we get

$$\begin{aligned}
&\parallel u_h^m-P_hu(t^m)\parallel_{L^2(\Omega)}+\Big(\sum_{n=0}^{m-1}\parallel(u_h^{n+1}-P_hu(t^{n+1}))-(u_h^n-P_hu(t^n))\parallel^2_{L^2(\Omega)}\Big)^{1/2}\\
&\qquad+\sqrt{\nu}\Big(\sum_{n=0}^{m-1}k|u_h^{n+1}-P_hu(t^{n+1})|^2_{H^1(\Omega)}\Big)^{1/2}\le C(h+H^2+k).
\end{aligned}\tag{6.3}$$

Then, (6.1) follows readily from the above result and the $P_h$'s properties. $\square$

As a consequence, if $h = H^2$ and $h \sim k$, then

$$\begin{aligned}\sup_{0\le n\le N} \| u_h^n - u(t^n) \|_{L^2(\Omega)} &+ \Big( \sum_{n=0}^{N-1} \| (u_h^{n+1} - u(t^{n+1})) - (u_h^n - u(t^n)) \|_{L^2(\Omega)}^2 \Big)^{1/2} \\ &+ \sqrt{\nu} \Big( \sum_{n=0}^{N-1} k |u_h^{n+1} - u(t^{n+1})|_{H^1(\Omega)}^2 \Big)^{1/2} \le Ch.\end{aligned} \tag{6.4}$$

Finally, we consider the error of the pressure. As in Section 5, the pressure satisfies the following bound.

**Lemma 6.2.** *Under the assumptions of Theorem* 4.6 *and Theorem* 6.1, *let* $(u(t^{n+1}), p(t^{n+1}))$ *and* $(u_h^{n+1}, p_h^{n+1})$ *be the respective solutions of* (1.1)–(1.4) *and* (1.20)–(1.21). *We have*

$$\begin{aligned}\Big( \sum_{n=0}^{N-1} k \| p_h^{n+1} - r_h p(t^{n+1}) \|_{L^2(\Omega)}^2 \Big)^{1/2} \le \frac{1}{\beta^\star} \Big\{ S_2 \Big( \sum_{n=0}^{N-1} k \| \frac{(u_h^{n+1} - u(t^{n+1})) - (u_h^n - u(t^n))}{k} \|_{L^2(\Omega)}^2 \Big)^{1/2} \\ + C(H^2 + h + k) \Big\},\end{aligned} \tag{6.5}$$

*where* $\beta^\star$ *is the constant of the inf-sup condition* (1.10) *and the constant* $C$ *depends on* $u, u', p$ *et* $p'$ *but does not depend on* $H, h$ *and* $k$.

*Proof.* The only difference with the proof of Lemma 5.1 concerns the non-linear term. Here we write

$$\begin{aligned}u(s) \cdot \nabla u(s) - u_H^{n+1} \cdot \nabla u_h^{n+1} &= (u(s) - u_H^{n+1}) \cdot \nabla u(s) + (u_H^{n+1} - u(s)) \cdot \nabla (u(s) - u_h^{n+1}) \\ &\quad + u(s) \cdot \nabla (u(s) - u_h^{n+1}),\end{aligned}$$

and

$$\begin{aligned}\| \Big( u(s) \cdot \nabla u(s) - u_H^{n+1} \cdot \nabla u_h^{n+1}, w_h^{n+1} \Big) \|_{L^2(\Omega)} \le \Big\{ \| u(s) - u_H^{n+1} \|_{L^2(\Omega)} \| u(s) \|_{W^{1,4}(\Omega)} \\ + \Big( \| u_H^{n+1} - u(s) \|_{L^4(\Omega)} + \| u(s) \|_{L^4(\Omega)} \Big) |u(s) - u_h^{n+1}|_{H^1(\Omega)} \Big\} \| w_h^{n+1} \|_{L^4(\Omega)} .\end{aligned}$$

Let us estimate the terms that compose the non-linear term. We have

$$\begin{aligned}&\Big| \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \Big( (u(s) - u_H^{n+1}).\nabla u(s), w_h^{n+1} \Big) ds \Big| \\ &\le S_4 (\sup_s \| u(s) \|_{W^{1,4}(\Omega)}) \Big( \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \| u(s) - u_H^{n+1} \|_{L^2(\Omega)}^2 \, ds \Big)^{1/2} \Big( \sum_{n=0}^{N-1} k |w_h^{n+1}|_{H^1(\Omega)}^2 \Big)^{1/2} \\ &\le (C_1 k \| u' \|_{L^2(\Omega \times ]0,T[)^2} + C(H^2 + k)) \Big( \sum_{n=0}^{N-1} k |w_h^{n+1}|_{H^1(\Omega)}^2 \Big)^{1/2}.\end{aligned}$$

Similarly, the second term is bounded as follows :

$$\Big|\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\Big((u_H^{n+1}-u(s)).\nabla(u(s)-u_h^{n+1})+u(s).\nabla(u(s)-u_h^{n+1}),w_h^{n+1}\Big)ds\Big|$$
$$\le S_4\Big(\sup_n \parallel u_H^{n+1}-u(s)\parallel_{L^4(\Omega)}+\sup_s\parallel u(s)\parallel_{L^4(\Omega)}\Big)\Big(\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}|u(s)-u_h^{n+1}|^2_{H^1(\Omega)}ds\Big)^{1/2}\Big(\sum_{n=0}^{N-1}k|w_h^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2}$$
$$\le C_2(H^2+k+h)\Big(\sum_{n=0}^{N-1}k|w_h^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2}.$$

Then, (6.5) follows readily from these bounds and from the inf-sup condition (1.10). □

Therefore, here again, we must derive an estimate for

$$\Big(\sum_{n=0}^{N-1}k\parallel\frac{(u_h^{n+1}-u(t^{n+1}))-(u_h^n-u(t^n))}{k}\parallel^2_{L^2(\Omega)}\Big)^{1/2}.$$

**Lemma 6.3.** *Under the assumptions of Theorem* 4.6 *and Theorem* 6.1, *and if* $\nabla u\in L^\infty(\Omega\times]0,T[)^2$, $\Delta u'\in L^2(\Omega\times]0,T[)^2$ *and* $\nabla p'\in L^2(\Omega\times]0,T[)$, *there exists a constant* $C$ *that does not depend on* $H,h$ *and* $k$, *such that*

$$\Big(\sum_{n=0}^{N-1}k\parallel\frac{(u_h^{n+1}-u(t^{n+1}))-(u_h^n-u(t^n))}{k}\parallel^2_{L^2(\Omega)}\Big)^{1/2}+\sqrt{\nu}\sup_{0\le n\le N}|u_h^n-u(t^n)|_{H^1(\Omega)}$$
$$+\sqrt{\nu}\Big(\sum_{n=0}^{N-1}|(u_h^{n+1}-u(t^{n+1}))-(u_h^n-u(t^n))|^2_{H^1(\Omega)}\Big)^{1/2}\le C(h+H^2+k). \tag{6.6}$$

*Proof.* As in the proof of Lemma 6.2, we insert $S_hu(t^{n+1})$, we set $w_h^n=u_h^n-S_hu(t^n)$ and we take

$$v_h^{n+1}=\frac{1}{k}(w_h^{n+1}-w_h^n)=\frac{1}{k}((u_h^{n+1}-S_hu(t^{n+1}))-(u_h^n-S_hu(t^n))).$$

Thus we obtain

$$\frac{1}{k^2}\parallel w_h^{n+1}-w_h^n\parallel^2_{L^2(\Omega)}+\frac{\nu}{k}(\nabla w_h^{n+1},\nabla(w_h^{n+1}-w_h^n))$$
$$=\frac{1}{k^2}\Big((S_hu(t^{n+1})-u(t^{n+1}))-(S_hu(t^n)-u(t^n)),w_h^{n+1}-w_h^n\Big)$$
$$+\int_{t^n}^{t^{n+1}}\frac{\nu}{k^2}\Big(\nabla(S_hu(t^{n+1})-S_hu(s)),\nabla(w_h^{n+1}-w_h^n)\Big)ds \tag{6.7}$$
$$+\frac{1}{k^2}\int_{t^n}^{t^{n+1}}\Big(u(s)\cdot\nabla u(s)-u_H^{n+1}\cdot\nabla u_h^{n+1},w_h^{n+1}-w_h^n\Big)ds.$$

Then by multiplying (6.7) by $k$ and by summing over $n=0,...,m-1$, we obtain the following left-hand side

$$\sum_{n=0}^{m-1}k\parallel\frac{w_h^{n+1}-w_h^n}{k}\parallel^2_{L^2(\Omega)}+\frac{\nu}{2}\sum_{n=0}^{m-1}\parallel\nabla(w_h^{n+1}-w_h^n)\parallel^2_{L^2(\Omega)}+\frac{\nu}{2}\Big(\parallel\nabla w_h^m\parallel^2_{L^2(\Omega)}-\parallel\nabla w_h^0\parallel^2_{L^2(\Omega)}\Big).$$

Let us bound the right-hand side of (6.7). The first term is bounded as follows : For any $\varepsilon_1 > 0$, we have

$$
\begin{aligned}
&\Big|\sum_{n=0}^{m-1} k\Big(\frac{(S_h u(t^{n+1}) - u(t^{n+1})) - (S_h u(t^n) - u(t^n))}{k}, \frac{w_h^{n+1} - w_h^n}{k}\Big)\Big| \\
&\leq \frac{1}{2}\Big(\frac{Ch^2}{\varepsilon_1}\Big( \| u' \|^2_{L^2(0,T;H^1(\Omega)^2)} + \| p' \|^2_{L^2(\Omega\times]0,T[)} \Big) + \varepsilon_1 \sum_{n=0}^{N-1} k \, \| \frac{w_h^{n+1} - w_h^n}{k} \|^2_{L^2(\Omega)} \Big).
\end{aligned}
$$

The second term is treated as follows : For any $\varepsilon_2 > 0$, we have

$$
\begin{aligned}
&\Big|\nu \sum_{n=0}^{m-1} \int_{t^n}^{t^{n+1}} \Big(\nabla\Big(\int_s^{t^{n+1}} \frac{d}{d\tau}(S_h(u))d\tau\Big), \nabla\Big(\frac{w_h^{n+1} - w_h^n}{k}\Big)\Big)ds\Big| \\
&\leq \nu\Big| \sum_{n=0}^{m-1} - \int_{t^n}^{t^{n+1}} \Big(\Delta u'(s), \frac{w_h^{n+1} - w_h^n}{k}\Big)(s - t^n)ds + \sum_{n=0}^{m-1} \int_{t^n}^{t^{n+1}} \Big(\nabla p'(s), \frac{w_h^{n+1} - w_h^n}{k}\Big)(s - t^n)ds\Big| \\
&\leq \frac{C\nu k^2}{2\sqrt{3}\varepsilon_2}\Big( \| \Delta u' \|_{L^2(\Omega\times]0,T[)^2} + \| \nabla p' \|_{L^2(\Omega\times]0,T[)} \Big) + \frac{\varepsilon_2 \nu}{2\sqrt{3}} \sum_{n=0}^{m-1} k \, \| \frac{w_h^{n+1} - w_h^n}{k} \|^2_{L^2(\Omega)} .
\end{aligned}
$$

For the third term, we set $\| \nabla u(t^{n+1}) \|_{L^\infty(\Omega)} \leq C$, and for any $\varepsilon_3 > 0$, we have

$$
\begin{aligned}
&\Big|\sum_{n=0}^{m-1} k\Big((u(t^{n+1}) - u_H^{n+1}).\nabla u(t^{n+1}), \frac{w_h^{n+1} - w_h^n}{k}\Big)\Big| \\
&\leq \frac{C}{2\varepsilon_3}(H^4 + k^2) + \frac{C\varepsilon_3}{2} \sum_{n=0}^{m-1} k \, \| \frac{w_h^{n+1} - w_h^n}{k} \|^2_{L^2(\Omega)} .
\end{aligned}
$$

For the fourth term, we use the fact that $\| u(t) \|_{L^\infty(\Omega)} \leq C$ and $|u_H^n|_{W^{1,5/2}(\Omega)} \leq C$ which means that $\| u_H^n \|_{L^\infty(\Omega)} \leq C$, and we apply (6.1). Then, for any $\varepsilon_4 > 0$, we have

$$
\begin{aligned}
&\Big|\sum_{n=0}^{m-1} k\Big((u_H^{n+1} - u(t^{n+1})).\nabla(u(t^{n+1}) - u_h^{n+1}), \frac{w_h^{n+1} - w_h^n}{k}\Big)\Big| \\
&\leq \frac{1}{2\varepsilon_4}C(H^4 + h^2 + k^2) + \frac{\varepsilon_4}{2} \sum_{n=0}^{N-1} k \, \| \frac{w_h^{n+1} - w_h^n}{k} \|^2_{L^2(\Omega)} .
\end{aligned}
$$

The fifth term is bounded as the fourth term. For any $\varepsilon_5 > 0$, we have

$$
\begin{aligned}
&\Big|\sum_{n=0}^{m-1} k\Big(u(t^{n+1}) \cdot \nabla(u(t^{n+1}) - u_h^{n+1}), \frac{w_h^{n+1} - w_h^n}{k}\Big)\Big| \\
&\leq \frac{C(\sup_n \| u(t^n) \|_{L^\infty(\Omega)})}{2\varepsilon_5}(H^2 + h + k)^2 + \frac{\varepsilon_5}{2} \sum_{n=0}^{m-1} k \, \| \frac{w_h^{n+1} - w_h^n}{k} \|^2_{L^2(\Omega)} .
\end{aligned}
$$

The last term is split into two parts that we treat successively.

$$
\begin{aligned}
&\Big|\sum_{n=0}^{m-1} \int_{t^n}^{t^{n+1}} \Big(u(s) \cdot \nabla u(s) - u(t^{n+1}) \cdot \nabla u(t^{n+1}), \frac{w_h^{n+1} - w_h^n}{k}\Big)ds\Big| \\
&= \Big|\sum_{n=0}^{m-1} \int_{t^n}^{t^{n+1}} \Big((u(s) - u(t^{n+1})) \cdot \nabla u(s) + u(t^{n+1}) \cdot \nabla(u(s) - u(t^{n+1})), \frac{w_h^{n+1} - w_h^n}{k}\Big)ds\Big|
\end{aligned}
$$

The first part is bounded as follows : For any $\varepsilon_6 > 0$, we have

$$\begin{aligned}
&\Big|\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\Big((u(s)-u(t^{n+1}))\cdot\nabla u(s), \frac{w_h^{n+1}-w_h^n}{k}\Big)ds\Big| \\
&=\Big|-\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\Big((\int_{t^n}^{\tau}u'(\tau)\cdot\nabla u(s)ds)d\tau, \frac{w_h^{n+1}-w_h^n}{k}\Big)\Big| \\
&\leq \frac{\| u' \|_{L^\infty(0,T;L^4(\Omega)^2)}}{3\varepsilon_6}\| \nabla u \|^2_{L^2(0,T;L^4(\Omega)^2)}\, k^2+\varepsilon_6\sum_{n=0}^{m-1}k\,\| \frac{w_h^{n+1}-w_h^n}{k}\|^2_{L^2(\Omega)},
\end{aligned}$$

and the second part is bounded as follows : For any $\varepsilon_7 > 0$, we have

$$\begin{aligned}
&\Big|\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\Big(u(t^{n+1})\cdot\nabla(u(s)-u(t^{n+1})), \frac{w_h^{n+1}-w_h^n}{k}\Big)ds\Big| \\
&\leq \| u \|_{L^\infty(\Omega\times]0,T[)}\Big\{\frac{\varepsilon_7}{2}\sum_{n=0}^{m-1}k\,\| \frac{w_h^{n+1}-w_h^n}{k}\|^2_{L^2(\Omega)}+\frac{k^2}{2\varepsilon_7}\| u' \|_{L^2(0,T;H^1(\Omega)^2)}\Big\}.
\end{aligned}$$

Then (6.6) follows readily after a suitable choice of $\varepsilon_i$ and by applying the $S_h's$ properties. $\square$

These two lemmas yield immediately the following theorem.

**Theorem 6.4.** *Under the assumptions*

$$u\in L^2(0,T;H^2(\Omega)^2)\cap\ L^\infty(0,T;H^1(\Omega)^2), u'\in L^2(0,T;H^2(\Omega)^2), p\in L^\infty(0,T;H^1(\Omega)),$$
$$p'\in L^2(0,T;H^1(\Omega)), f\in L^2(\Omega\times]0,T[)^2, f'\in L^2(0,T;H^{-1}(\Omega)^2), f(0)\in L^2(\Omega)^2$$

*and $\Omega$ convex, we have*

$$\Big(\sum_{n=0}^{N-1}k\,\| p(t^{n+1})-p_h^{n+1}\|^2_{L^2(\Omega)}\Big)^{1/2}\leq C(h+H^2+k), \tag{6.8}$$

*with a constant $C$ that does not depend on $h, H$ and $k$.*

**Remark 6.5.** *As a consequence, if $h = H^2$ and $h\sim k$, then*

$$\Big(\sum_{n=0}^{N-1}k\,\| p(t^{n+1})-p_h^{n+1}\|^2_{L^2(\Omega)}\Big)^{1/2}\leq Ch. \tag{6.9}$$

This analysis is confirmed by numerical results which are presented in the next section.

## 7. Numerical results

In order to confirm these results numerically, we did several experiments by using the FreeFem ++ software, see [9] for the “mini” element.
On the square domain $]0,1[\times]0,1[$, the numerical velocity and the pressure are taken as $(u,p) = (\text{curl }\ \psi, p)$, where:

$$\psi(t,x,y) = te^{-t^2(x+y)}y^2(1-y)^2\sin^2(\pi x),$$

and

$$p(t,x,y) = te^{-t}\cos(2\pi x)\sin(2\pi y).$$

First of all, we have verified that our problem is stable. In fact, we have fixed the coarse grid $N_g = 7$ points, so the fine one contains $N_f = N_g^2 = 49$ points and $T = 500$ so that the number of iterations becomes $nbiter = T\times N_f$.

We present below the evolution of the degree of liberty 500 in time. The graphs are as follows :

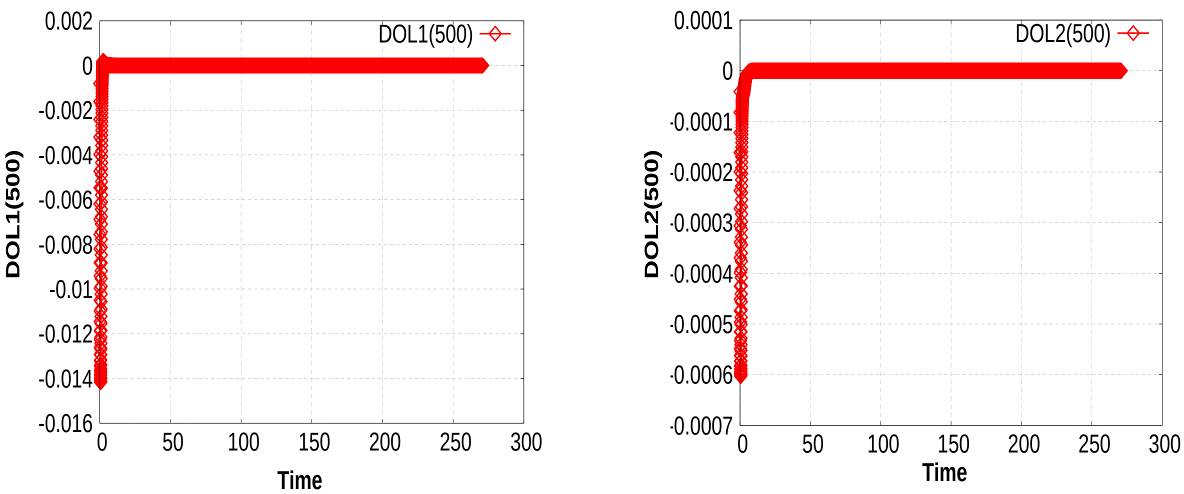


FIGURE 1. The first figure shows the evolution of the first component of the degree of liberty 500 in time and the second one is related to the second component.

In order to see precisely this evolution, we did a zoom on the velocity's components. We obtain :

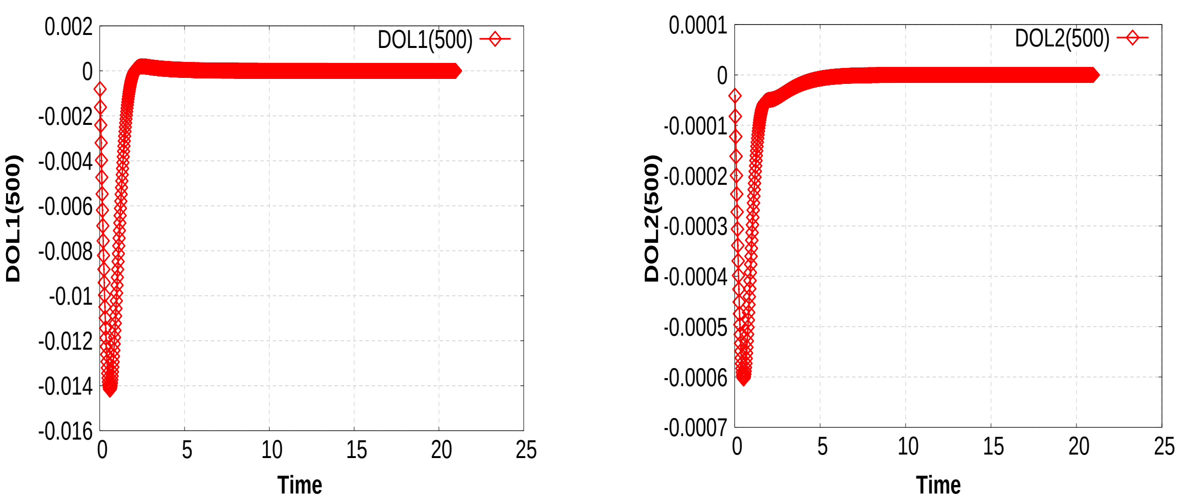


FIGURE 2. The first figure shows a zoom on the evolution of the first component of the degree of liberty 500 in time and the second one is related to the second component.

Next, we have taken $N_g = 10, N_f = N_g^2, h = \dfrac{1}{N_f}, T = 1$ and $nbiter = T \times N_f$ and we have obtained a color comparison between the exact and numerical solutions of velocity and pressure :

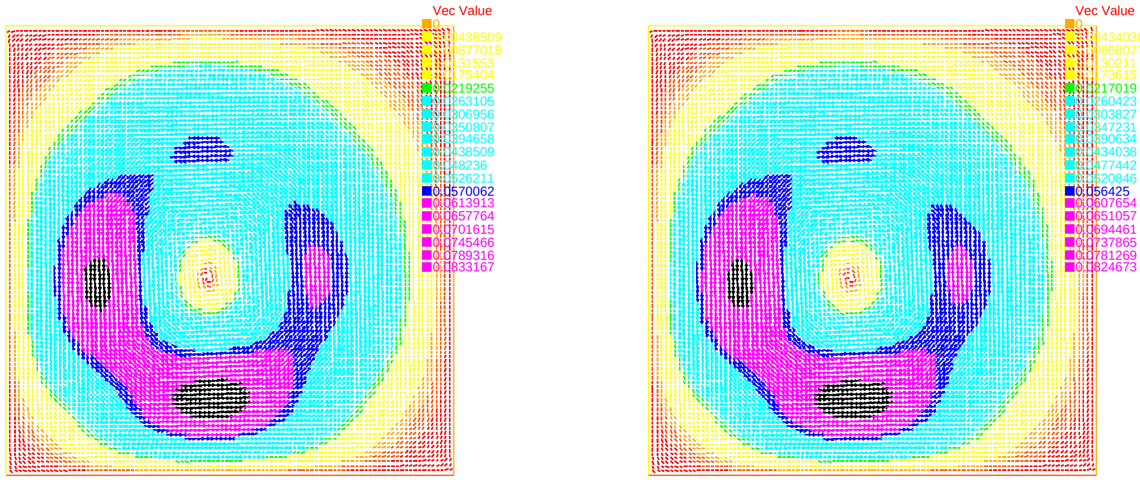


FIGURE 3. These two figures show respectively the exact and numerical velocity's solutions.

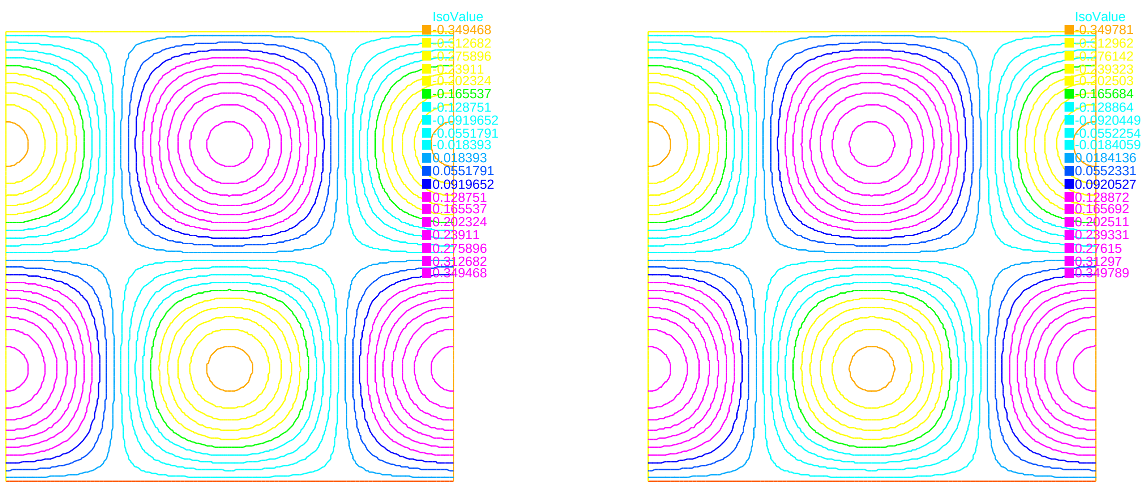


FIGURE 4. These two figures show respectively the exact and numerical pressure's solutions.

The graphs related to the velocity's and pressure's error estimations have been studied. The values of these error estimations are given by the following table.

| meshes | $N_f$ | $L^2$ Rate | $H^1$ Rate | $L^2$ pressure Rate |
|---|---|---|---|---|
| $H = 1/4, h = 1/16$ | 16 | $-2.42327$ | -0.578102 | -2.43816 |
| $H = 1/6, h = 1/36$ | 36 | $-2.87689$ | -1.17702 | -3.04758 |
| $H = 1/8, h = 1/64$ | 64 | $-3.1673$ | -1.47656 | -3.55464 |
| $H = 1/10, h = 1/100$ | 100 | $-3.37502$ | -1.63477 | -3.9356 |

So the $L^2(\Omega\times]0,T[)^2$ slope is of order 1.1958 and the $L^2(0,T;H^1(\Omega)^2)$ slope is of order 1.327 and the pressure's one in norm $L^2(\Omega\times]0,T[)$ is of order 1.8814.

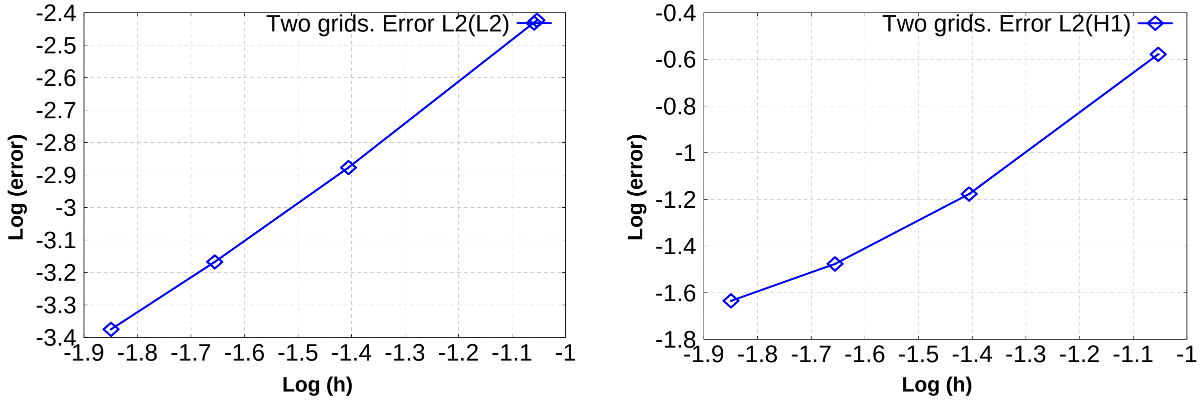


FIGURE 5. The first figure shows the error $L^2_{t,x}$ and the second one shows the error $L^2_t(H^1_x)$.

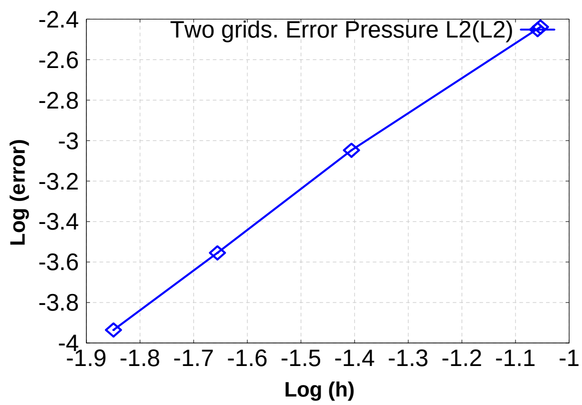


FIGURE 6. The pressure's error in norm $L^2_{t,x}$.

**Remark 7.1.** An explicit scheme on the coarse grid has been studied numerically and theorically. The results are interesting.

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